DYNAMICAL MULTIFRACTAL ZETA-FUNCTIONS AND FINE MULTIFRACTAL SPECTRA OF GRAPH-DIRECTED SELF-CONFORMAL CONSTRUCTIONS

V. Mijović
Department of Mathematics
University of St. Andrews
St. Andrews, Fife KY16 9SS, Scotland
e-mail: vm27@st-and.ac.uk

L. Olsen
Department of Mathematics
University of St. Andrews
St. Andrews, Fife KY16 9SS, Scotland
e-mail: lo@st-and.ac.uk

Abstract. We introduce multifractal pressure and dynamical multifractal zeta-functions providing precise information of a very general class of multifractal spectra, including, for example, the fine multifractal spectra of graph-directed self-conformal measures and the fine multifractal spectra of ergodic Birkhoff averages of continuous functions on graph-directed self-conformal sets.

Contents

1. Introduction.

For a Borel measure μ on $\mathbb{R}^d$ and a positive number α, let us consider the set of those points x in $\mathbb{R}^d$ for which the measure $\mu(B(x,r))$ of the ball $B(x,r)$ with center x and radius r behaves like r^α for small r, i.e. the set

$$\left\{ x \in K \,\middle|\, \lim_{r\searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}. \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary: 28A78. Secondary: 37D30, 37A45.
Key words and phrases: multifractals, zeta functions. pressure, Bowen's formula, large deviations, Hausdorff dimension, graph-directed self-conformal sets

If the intensity of the measure μ varies very widely, it may happen that the sets in (1.1) display a fractal-like character for a range of values of α. In this case it is natural to study the Hausdorff dimensions of the sets in (1.1) as α varies. We therefore define the fine multifractal spectrum of μ by

$$f_\mu(\alpha)=\dim_{\mathsf{H}}\left\{x\in K\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\right\}. \tag{1.2}$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension; here and below we use the following convention, namely, we define the Hausdorff of the empty set to be $-\infty$, i.e. we put

$$\dim_{\mathsf{H}}\varnothing=-\infty\,.$$

The fine multifractal spectrum is one of the the two main ingredients in multifractal analysis. The second main ingredient is the Renyi dimensions. Renyi dimensions quantify the varying intensity of a measure by analyzing its moments at different scales. Formally, for $q\in\mathbb{R}$, the q'th Renyi dimensions $\tau_\mu(q)$ of μ is defined by

$$\tau_\mu(q)=\lim_{r\searrow 0}\frac{\log\displaystyle\int_K\mu(B(x,r))^{q-1}\,d\mu(x)}{-\log r}\,,$$

provided the limit exists. One of the main problems in multifractal analysis is to understand the multifractal spectrum and the Renyi dimensions, and their relationship with each other. During the past 20 years there has been an enormous interest in computing the multifractal spectra of measures in the mathematical literature and within the last 15 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, see the textbooks [Fa,Pe] and the references therein.

Dynamical zeta-functions were introduced by Artin & Mazur in the mid 1960's [ArMa] based on an analogy with the number theoretical zeta-functions associated with a function field over a finite ring. Subsequently Ruelle [Rue1,Rue2] associated zeta-functions to certain statistical mechanical models in one dimensions. During the past 35 years many parallels have been drawn between number theory zeta-functions, dynamical zeta-functions, and statistical mechanics zeta-functions. However, much more recently and motivated by the powerful techniques provided by the use of the Artin-Mazur zeta-functions in number theory and the use of the Ruelle zeta-functions in dynamical systems, Lapidus and collaborators (see the intriguing books by Lapidus & van Frankenhuysen [Lap-vF1,Lap-VF2] and the references therein) have recently introduced and pioneered to use of zeta-functions in fractal geometry. Inspired by this development, within the past 4–5 years several authors have paralleled this development by introducing zeta-functions into multifractal geometry. Indeed, in 2009, Lapidus and collaborators introduced various intriguing *geometric* multifractal zeta-functions [LapRo,LapLe-VeRo] designed to provide information about the multifractal spectrum $f_\mu(\alpha)$ of self-similar measures μ, and many connections with multifractal spectra were suggested and in some cases proved; for example, in simplified cases the multifractal spectrum of a self-similar measure could be recovered from a zeta-function. The key idea in [LapRo,LapLe-VeRo] is both simple and attractive: while traditional zeta-functions are defined by "summing over all data", the multifractal zeta-functions in [LapRo,LapLe-VeRo] are defined by only "summing over data that are multifractally relevant". This idea is also the leitmotif in this work (as well as in earlier work [Bak,MiOl,Ol4]), see, in particular, the first remark following the definition of the zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ in Section 4. Ideas similar to those in [LapRo,LapLe-VeRo] have very recently been revisited and investigated in [Bak,MiOl] where the authors introduce related *geometric* multifractal zeta-functions tailored to study the multifractal spectra of self-conformal measures and a number of connections with very general types of multifractal spectra were established.

We also point out that the work by Lapidus et al. [LapRo,LapLe-VeRo] was followed shortly afterwards by the introduction of a different type of multifractal zeta-function by Levy-Vehel & Mendivil [Le-VeMe] tailored to provide information about the Renyi dimensions $\tau_\mu(q)$ of self-similar measures μ. Ideas related to those in [Le-VeMe] have also been investigated in [Ol2,Ol3] where the author introduce multifractal zeta-functions designed to study the Renyi dimensions and the (closely related) multifractal Minkowski volume of self-conformal measures.

In addition to the distinctively *geometric* approaches in [Bak,Le-VeMe,MiOl,Ol2,Ol3], it has been a major challenge to introduce and develop a natural and meaningful theory of *dynamical* multifractal zeta-functions paralleling the existing powerful theory of *dynamical* zeta-functions introduced and developed by Ruelle [Rue1,Rue2] and others, see, for example, the surveys and books [Bal1,Bal2,ParPo1,ParPo2] and the references therein. In particular, in the setting of self-conformal constructions, [Ol4] introduced a family of *dynamical* multifractal zeta-functions designed to provide precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions.

However, recently it has been recognised that while self-conformal constructions provide a useful and important framework for studying fractal and multifractal geometry, the more general notion of *graph-directed* self-conformal constructions provide a substantially more flexible and useful framework, see, for example, [MaUr] for an elaboration of this. In recognition of this viewpoint, the purpose of this paper is to developed a *dynamical* theory of multifractal zeta-functions in the setting of *graph-directed* self-conformal constructions.

In Section 2–3 we briefly recall the definitions of self-conformal constructions and the accompanying pressure and dynamical zeta-functions. In Section 4 we provide our main definitions of the multifractal pressure and the multifractal dynamical zeta-functions and we state our main results. Section 5 contains a number of examples, including, multifractal spectra of graph-directed self-conformal measures and different types of multifractal spectra of ergodic averages of continuous functions on graph-directed self-conformal sets. Finally, the proofs are presented in Sections 6–10.

2. The setting, Part 1: Graph-directed self-conformal sets and graph-directed self-conformal measures.

2.1. Notation from symbolic dynamics. We first recall the notation and terminology from symbolic dynamics that will be used in this paper. Fix a finite directed multigraph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ where V denotes the set of vertices of G and E denotes the set of edges of G. We will always assume that the graph G is strongly connected. For an edge $\mathsf{e} \in \mathsf{E}$, we write $\mathrm{i}(\mathsf{e})$ for the initial vertex of e and we write $\mathrm{t}(\mathsf{e})$ for the terminal vertex of e. For $\mathsf{i}, \mathsf{j} \in \mathsf{V}$, write

$$\begin{aligned} \mathsf{E}_{\mathsf{i}} &= \left\{ \mathsf{e} \in \mathsf{E} \,\middle|\, \mathrm{i}(\mathsf{e}) = \mathsf{i} \right\}, \\ \mathsf{E}_{\mathsf{i},\mathsf{j}} &= \left\{ \mathsf{e} \in \mathsf{E} \,\middle|\, \mathrm{i}(\mathsf{e}) = \mathsf{i} \text{ and } \mathrm{t}(\mathsf{e}) = \mathsf{j} \right\}; \end{aligned} \tag{2.1}$$

i.e. E_{i} is the family of all edges starting at i; and $\mathsf{E}_{\mathsf{i},\mathsf{j}}$ is the is the family of all edges starting at i and ending at j. Also, for a positive integer n, we write

$$\begin{aligned} \Sigma_{\mathsf{G}}^{n} = \Big\{ \mathsf{e}_1 \dots \mathsf{e}_n \,\Big|\, & \mathsf{e}_i \in \mathsf{E} \text{ for } 1 \leq i \leq n, \\ & \mathrm{t}(\mathsf{e}_1) = \mathrm{i}(e_2), \\ & \mathrm{t}(e_{i-1}) = \mathrm{i}(\mathsf{e}_i) \text{ and } \mathrm{t}(\mathsf{e}_i) = \mathrm{i}(e_{i+1}) \text{ for } 1 < i < n, \\ & \mathrm{t}(e_{n-1}) = \mathrm{i}(\mathsf{e}_n) \Big\} \\ \Sigma_{\mathsf{G}}^{*} = \bigcup_{m} \Sigma_{\mathsf{G}}^{m}, & \\ \Sigma_{\mathsf{G}}^{\mathbb{N}} = \Big\{ \mathsf{e}_1 \mathsf{e}_2 \dots \,\Big|\, & \mathsf{e}_i \in \mathsf{E} \text{ for } 1 \leq i, \\ & \mathrm{t}(\mathsf{e}_1) = \mathrm{i}(e_2), \\ & \mathrm{t}(e_{i-1}) = \mathrm{i}(\mathsf{e}_i) \text{ and } \mathrm{t}(\mathsf{e}_i) = \mathrm{i}(e_{i+1}) \text{ for } 1 < i \Big\}; \end{aligned} \tag{2.2}$$

i.e. Σ_{G}^{n} is the family of all finite strings $\mathbf{i} = \mathsf{e}_1 \dots \mathsf{e}_n$ consisting of finite paths in G of length n; Σ_{G}^{*} is the family of all finite strings $\mathbf{i} = \mathsf{e}_1 \dots \mathsf{e}_m$ with $m \in \mathbb{N}$ consisting of finite paths in G; and $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ is the family of all infinite strings $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \dots$ consisting of infinite paths in G. For a finite string $\mathbf{i} = \mathsf{e}_1 \dots \mathsf{e}_n \in \Sigma_{\mathsf{G}}^{n}$, we write

$$\mathrm{i}(\mathbf{i}) = \mathrm{i}(\mathsf{e}_1), \quad \mathrm{t}(\mathbf{i}) = \mathrm{t}(\mathsf{e}_n), \tag{2.3}$$

and for an infinite string $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2\ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$, we write

$$\mathrm{i}(\mathbf{i}) = \mathrm{i}(\mathsf{e}_1)\,. \tag{2.4}$$

Next, for an infinite string $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2\ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ and a positive integer n, we will write $\mathbf{i}|n = \mathsf{e}_1\ldots\mathsf{e}_n$. In addition, for a positive integer n and a finite string $\mathbf{i} = \mathsf{e}_1\ldots\mathsf{e}_n \in \Sigma_{\mathsf{G}}^{n}$ with length equal to n, we will write $|\mathbf{i}| = n$, and we let $[\mathbf{i}]$ denote the cylinder generated by $\mathbf{i}$, i.e.

$$[\mathbf{i}] = \Big\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}\,\Big|\,\mathbf{j}|n = \mathbf{i}\Big\}\,. \tag{2.5}$$

Finally, let $S : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \Sigma_{\mathsf{G}}^{\mathbb{N}}$ denote the shift map, i.e.

$$S(\mathsf{e}_1\mathsf{e}_2\ldots) = \mathsf{e}_2\mathsf{e}_3\ldots\,.$$

2.2. Graph-directed self-conformal sets and graph-directed self-conformal measures. Next, we recall the definition of graph-directed self-conformal sets and measures. A graph-directed conformal iterated function system with probabilities is a list

$$\Big(\mathsf{V},\ \mathsf{E},\ (V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\ (X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\ (S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},\ (p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}\Big)$$

where

- For each $\mathsf{i} \in \mathsf{V}$ we have: V_{i} is an open, connected subset of $\mathbb{R}^d$.
- For each $\mathsf{i} \in \mathsf{V}$ we have: $X_{\mathsf{i}} \subseteq V_{\mathsf{i}}$ is a compact set with $X_{\mathsf{i}}^{\circ\,-} = X_{\mathsf{i}}$.
- For each $\mathsf{i},\mathsf{j} \in \mathsf{V}$ and $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ we have: $S_{\mathsf{e}} : V_{\mathsf{j}} \to V_{\mathsf{i}}$ is a contractive $C^{1+\gamma}$ diffeomorphism with $0 < \gamma < 1$ such that $S_{\mathsf{e}}(X_{\mathsf{j}}) \subseteq X_{\mathsf{i}}$.
- The Conformality Condition. For $\mathsf{i},\mathsf{j} \in \mathsf{V}$, $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ and $x \in V_{\mathsf{j}}$, let

  $$(DS_{\mathsf{e}})(x) : \mathbb{R}^d \to \mathbb{R}^d$$

  denote the derivative of S_{e} at x. For each $\mathsf{i},\mathsf{j} \in \mathsf{V}$ and $\mathsf{e} \in E_{\mathsf{i},\mathsf{j}}$ we have: $(DS_{\mathsf{e}})(x)$ is a contractive similarity map, i.e. there exists $s_{\mathsf{e}}(x) \in (0,1)$ such that $|(DS_{\mathsf{e}})(x)u - (DS_{\mathsf{e}})(x)v| = s_{\mathsf{e}}(x)|u-v|$ for all $u, v \in \mathbb{R}^d$.
- For each $\mathsf{i} \in \mathsf{V}$ we have: $(p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}_{\mathsf{i}}}$ is a probability vector.

It follows from [Hu] that there exists a unique list $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of non-empty compact sets $K_{\mathsf{i}} \subseteq X_{\mathsf{i}}$ such that

$$K_{\mathsf{i}} = \bigcup_{\mathsf{e}\in\mathsf{E}_{\mathsf{i}}} S_{\mathsf{e}}K_{\mathrm{t}(\mathsf{e})}\,, \tag{2.6}$$

and a unique list $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of probability measures with $\operatorname{supp}\mu_{\mathsf{i}} = K_{\mathsf{i}}$ such that

$$\mu_{\mathsf{i}} = \sum_{\mathsf{e}\in E_{\mathsf{i}}} p_{\mathsf{e}}\,\mu_{\mathrm{t}(\mathsf{e})}\circ S_{\mathsf{e}}^{-1}\,. \tag{2.7}$$

The sets $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ and measures $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ are called the self-conformal sets and self-conformal measures associated with the list $(\mathsf{V},\ \mathsf{E},\ (V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\ (X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\ (S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},\ (p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}})$, respectively. We will frequently assume that the so-called Open Set condition (OSC) is satisfied. The OSC is defined as follows:

- The Open Set Condition: There exists a list $(U_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of open non-empty and bounded sets $U_{\mathsf{i}} \subseteq X_{\mathsf{i}}$ with $S_{\mathsf{e}}\big(U_{\mathsf{j}}\big) \subseteq U_{\mathsf{i}}$ for all $\mathsf{i},\mathsf{j} \in \mathsf{V}$ and all $\mathsf{e} \in \mathsf{E}_{\mathsf{i},\mathsf{j}}$ such that $S_{\mathsf{e}_1}\big(U_{\mathrm{t}(\mathsf{e}_1)}\big) \cap S_{\mathsf{e}_2}\big(U_{\mathrm{t}(\mathsf{e}_2)}\big) = \varnothing$ for all $\mathsf{i} \in \mathsf{V}$ and all $\mathsf{e}_1, \mathsf{e}_2 \in \mathsf{E}_{\mathsf{i}}$ with $\mathsf{e}_1 \neq \mathsf{e}_2$.

For $\mathbf{i} = \mathsf{e}_1 \dots \mathsf{e}_n \in \Sigma_{\mathsf{G}}^n$, we write

$$\begin{aligned} S_{\mathbf{i}} &= S_{\mathsf{e}_1} \cdots S_{\mathsf{e}_n} \,, \\ K_{\mathbf{i}} &= S_{\mathsf{e}_1} \cdots S_{\mathsf{e}_n} \big(K_{\mathsf{t}(\mathsf{e}_n)} \big) \,, \\ p_{\mathbf{i}} &= p_{\mathsf{e}_1} \cdots p_{\mathsf{e}_n} \,, \end{aligned} \tag{2.8}$$

and we define the projection $\pi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^d$ by

$$\{ \, \pi(\mathbf{i}) \, \} = \bigcap_n K_{\mathbf{i}|n} \tag{2.9}$$

for $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$. Finally, we define $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ by

$$\Lambda(\mathbf{i}) \,=\, \log \big| \big(DS_{\mathsf{e}_1} \big) \big(\pi_{\mathsf{t}(\mathsf{e}_1)}(S\mathbf{i}) \big) \big| \tag{2.10}$$

for $\mathbf{i} = \mathsf{e}_1 \mathsf{e}_2 \ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$; loosely speaking the map Λ represents the local change of scale as one goes from $\pi_{\mathsf{t}(\mathsf{e}_1)}(S\mathbf{i})$ to $\pi_{\mathsf{i}(\mathsf{e}_1)}(\mathbf{i})$.

3. The setting, Part 2: Pressure and dynamical zeta-functions.

Throughout this section, and in the remaining parts of the paper, we will used the following notation. Namely, if $(a_n)_n$ is a sequence of complex numbers and if f is the power series defined by $f(z) = \sum_n a_n z^n$ for $z \in \mathbb{C}$, then we will denote the radius of convergence of f by $\sigma_{\mathsf{rad}}(f)$, i.e. we write

$$\sigma_{\mathsf{rad}}(f) = \text{ “the radius of convergence of } f\text{” } .$$

Our definitions and results are motivated by the notion of pressure from the thermodynamic formalism and the dynamical zeta-functions introduced by Ruelle [Rue1,Rue2]; see, also [Bal1,Bal2, ParPo1,ParPo2]. In addition, Bowen's formula expressing the Hausdorff dimension of a self-conformal set in terms of the pressure (or the dynamical zeta-function) of the scaling map Λ in (2.5) also plays a leitmotif in our work. Because of this we now recall the definition of pressure and dynamical zeta-function, and the statement of Bowen's formula. Let $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be a continuous function; here, and below, we equip $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with the product topology with discrete factors and all statements about continuity involving $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ will always refer to this topology. The pressure of φ is defined by

$$P(\varphi) = \lim_n \, \frac{1}{n} \, \log \sum_{\mathbf{i} \in \Sigma_{\mathsf{G}}^n} \, \sup_{\mathbf{u} \in [\mathbf{i}]} \, \exp \, \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \,, \tag{3.1}$$

see [Bo2] or [ParPo2]; we note that it is well-known that the limit in (3.1) exists. Also, the dynamical zeta-function of φ is defined by

$$\zeta^{\mathsf{dyn}}(\varphi; z) = \sum_n \, \frac{z^n}{n} \left(\sum_{\mathbf{i} \in \Sigma_{\mathsf{G}}^n} \, \sup_{\mathbf{u} \in [\mathbf{i}]} \, \exp \, \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right) \tag{3.2}$$

for those complex numbers z for which the series converge, see [ParPo2]. We now list two easily established and well-known properties of the pressure $P(\varphi)$ and of the radius of convergence $\sigma_{\mathsf{rad}}\big(\zeta^{\mathsf{dyn}}(\varphi; \cdot) \big)$ of the power-series $\zeta^{\mathsf{dyn}}(\varphi; \cdot)$. While both results are well-known and easily proved (see, for example, [Bar,Fa]), we have decided to list them since they play an important part in the discussion of our results.

Theorem A (see, for example, [Bar,Fa]). Radius of convergence. *Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. Then we have*

$$-\log \sigma_{\mathsf{rad}}\big(\zeta^{\mathsf{dyn}}(\varphi; \cdot) \big) = P(\varphi) \,.$$

Theorem B (see, for example, [Bar,Fa]). Continuity and monotony properties of the pressure. *Fix a continuous function* $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ *with* $\Phi < 0$. *Then the function* $t \to P(t\Phi)$, *where* $t \in \mathbb{R}$, *is continuous, strictly decreasing and convex with* $\lim_{t\to-\infty} P(t\Phi) = \infty$ *and* $\lim_{t\to\infty} P(t\Phi) = -\infty$. *In particular, there is a unique real number* s *such that*

$$P(s\Phi) = 0\,;$$

alternatively, s *is the unique real number such that*

$$\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn}}(s\Phi;\cdot)\,\big) = 1\,.$$

The main importance of the pressure (for the purpose of this exposition) is that it provides a beautiful formula for the Hausdorff dimension of a graph-directed self-conformal set satisfying the OSC. This result was first noted by [Bo1] (in the setting of quasi-circles) and is the content of the next result.

Theorem C (see, for example, [Bar,Fa]). Bowen's formula. *Let* $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ *be the graph-directed self-conformal sets defined by (2.6) and let* $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ *be the scaling function defined by (2.10). Let* s *be the unique real number such that*

$$P(s\Lambda) = 0\,;$$

alternatively, s *is the unique real number such that*

$$\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn}}(s\Lambda;\cdot)\,\big) = 1\,.$$

If the OSC is satisfied, then we have

$$\dim_{\mathsf{H}} K_{\mathsf{i}} = s$$

for all $\mathsf{i} \in \mathsf{V}$.

It is reasonable to expect that any meaningful theory of dynamical multifractal zeta-functions should produce multifractal analogues of Bowen's equation. In the next section we will develop such a theory for graph-directed self-conformal constructions (extending the theory for self-conformal constructions introduced in [Ol4]).

4. Statements of the main results.

We denote the family of Borel probability measures on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ and the family of shift invariant Borel probability measures on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ by $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, respectively, i.e. we write

$$\begin{aligned}
\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) &= \Big\{\mu \,\Big|\, \mu \text{ is a Borel probability measures on } \Sigma_{\mathsf{G}}^{\mathbb{N}} \Big\},\\
\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) &= \Big\{\mu \,\Big|\, \mu \text{ is a shift invariant Borel probability measures on } \Sigma_{\mathsf{G}}^{\mathbb{N}} \Big\};
\end{aligned}$$

we will always equip $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with the weak topologies. We now fix a metric space X and a continuous map $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$. The multifractal zeta-function framework developed in this paper depend on the space X and the map U; judicious choices of X and U will provide important examples, including, multifractal spectra of graph-directed self-conformal measures (see Section 5.1) and a variety of multifractal spectra of ergodic averages of continuous functions on graph-directed self-conformal sets (see Section 5.2). Next, for a positive integer n, let $L_n : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ be defined by

$$L_n\mathbf{i} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{S^k\mathbf{i}}\,; \tag{4.1}$$

recall, that $S : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \Sigma_{\mathsf{G}}^{\mathbb{N}}$ denotes the shift map. We can now define the multifractal pressure and zeta-function associated with the space X and the map U.

Definition. The multifractal pressure $\underline{P}_C^U(\varphi)$ and $\overline{P}_C^U(\varphi)$ associated with the space X and the map U. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. For $C \subseteq X$, we define the lower and upper mutifractal pressure of φ associated with the space X and the map U by*

$$\begin{aligned}
\underline{P}_C^U(\varphi) &= \liminf_n \frac{1}{n} \log \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ UL_n[\mathbf{i}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u}\,,\\
\overline{P}_C^U(\varphi) &= \limsup_n \frac{1}{n} \log \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ UL_n[\mathbf{i}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u}\,.
\end{aligned} \tag{4.2}$$

If $\underline{P}_C^U(\varphi)$ and $\overline{P}_C^U(\varphi)$ coincide, then we write $P_C^U(\varphi)$ for their common value, i.e. we write $P_C^U(\varphi) = \underline{P}_C^U(\varphi) = \overline{P}_C^U(\varphi)$.

Definition. The dynamical multifractal zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ associated with the space X and the map U. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. For $C \subseteq X$, we define the dynamical multifractal zeta-function $\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$ associated with the space X and the map U by*

$$\zeta_C^{\mathsf{dyn},U}(\varphi;z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ UL_n[\mathbf{i}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right) \tag{4.3}$$

for those complex numbers z for which the series converges.

Remark. Comparing the definition of the pressure (3.1) (the dynamical zeta-function (3.2)) and the definition of the multifractal pressure (4.2) (the dynamical multifractal zeta-function (4.3)), it is clear that the multifractal pressure (the dynamical multifractal zeta-function) is obtained by only summing over those strings $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ that are multifractally relevant, i.e. those strings $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ for which $UL_n[\mathbf{i}] \subseteq C$.

Remark. It is clear that if $C = X$, then the multifractal "constraint" $UL_n[\mathbf{i}] \subseteq C$ is vacuously satisfied, and that, in this case, the multifractal pressure and dynamical multifractal zeta-function reduce to the usual pressure and the usual dynamical zeta-function, i.e.

$$\underline{P}_X^U(\varphi) = \overline{P}_X^U(\varphi) = P(\varphi)$$

and

$$\zeta_X^{\mathsf{dyn},U}(\varphi;\cdot) = \zeta^{\mathsf{dyn}}(\varphi;\cdot)\,.$$

Before developing the theory of the multifractal pressure and the multifractal zeta-functions further we make to following two simple observations. Firstly, we note in Proposition 4.2 below that the expected relationship between the multifractal pressure and the radius of convergence of the multifractal zeta-function holds. Secondly, we would expect any dynamically meaningful theory of dynamical multifractal zeta-functions to lead to multifractal Bowen formulas. For this to hold, we must, at the very least, ensure that there are unique solutions to the relevant multifractal Bowen equations. i.e. we must ensure that if $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}}$ is a continuous function, then there is are unique real numbers $\mathit{f}(C)$ and $\mathcal{f}(C)$ solving the following multifractal Bowen equations, namely,

$$\begin{aligned}
\lim_{r\searrow 0} \overline{P}_{B(C,r)}^U(\,\mathit{f}(C)\,\Phi) &= 0\,,\\
\overline{P}_C^U(\,\mathcal{f}(C)\,\Phi) &= 0\,.
\end{aligned} \tag{4.4}$$

That there are unique numbers $\underline{f}(C)$ and $\overline{f}(C)$ satisfying (4.4) is our second preliminary observation, see Proposition 4.3. However, before stating and proving Proposition 4.2 and Proposition 4.3, we first show that the limit $\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(\,\underline{f}(C)\,\Phi)$ in (4.4) exists. This is the content of the next proposition.

Proposition 4.1. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a subset of X. Fix a continuous function $\varphi:\Sigma^{\mathbb{N}}\to\mathbb{R}$. Then the following limits*

$$\lim_{r\searrow 0}\underline{P}^{U}_{B(C,r)}(\varphi)\,,$$
$$\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(\varphi)\,,$$

exist.

Proof
This follows immediately from the fact that the function $r\to\overline{P}^{U}_{B(C,r)}(\varphi)$ is monotone. $\square$

We can now state Proposition 4.2 and Proposition 4.3.

Proposition 4.2. Radius of convergence. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a subset of X. Fix a continuous function $\varphi:\Sigma^{\mathbb{N}}\to\mathbb{R}$. We have*

$$-\log\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn},U}_{C}(\varphi;\cdot)\,\big)=\overline{P}^{U}_{C}(\varphi)\,.$$

Proof
This follows immediately from the fact that if $(a_n)_n$ is a sequence of complex numbers and if f denotes the power series defined by $f(z)=\sum_n a_nz^n$, then $\sigma_{\mathsf{rad}}(f)=\frac{1}{\limsup_n|a_n|^{\frac{1}{n}}}$. $\square$

Proposition 4.3. Continuity and monotonicity of the multifractal pressure. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a subset of X. Fix a continuous function $\Phi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ with $\Phi<0$. Then the functions $t\to\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(t\Phi)$ and $t\to\overline{P}^{U}_{C}(t\Phi)$, where $t\in\mathbb{R}$, are continuous, strictly decreasing and convex with $\lim_{t\to-\infty}\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(t\Phi)=\infty$ and $\lim_{t\to\infty}\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(t\Phi)=-\infty$, and $\lim_{t\to-\infty}\overline{P}^{U}_{C}(t\Phi)=\infty$ and $\lim_{t\to\infty}\overline{P}^{U}_{C}(t\Phi)=-\infty$. In particular, there are unique real numbers $\underline{f}(C)$ and $\overline{f}(C)$ such that*

$$\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(\,\underline{f}(C)\,\Phi)=0\,,\tag{4.5}$$
$$\overline{P}^{U}_{C}(\,\overline{f}(C)\,\Phi)=0\,;\tag{4.6}$$

alternatively, $\underline{f}(C)$ and $\overline{f}(C)$ are the unique real numbers such that

$$\lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta^{U}_{B(C,r)}(\,\underline{f}(C)\,\Phi;\cdot)\,\big)=1\,,$$
$$\sigma_{\mathsf{rad}}\big(\,\zeta^{U}_{C}(\,\overline{f}(C)\,\Phi;\cdot)\,\big)=1\,.$$

Proof
This statement is not difficult to prove and, for the sake of brevity, we have therefore decided to omit the proof. $\square$

We will now state our main results. The results are divided into two parts: the first part consists of Theorem 4.4 and Corollary 4.5, and the second part consists of Theorem 4.6 and Corollary 4.7. The motivation for this is the following. Let Λ denote the scaling map in (2.10). For judicious choices of X and U, we are clearly attempting to relate the solutions $\underline{f}(C)$ and $\overline{f}(C)$ of the multifractal Bowen equations (4.5) and (4.6) to various multifractal spectra. The following simple example serves

to illustrate this. Namely, let $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the graph-directed self-conformal measures in (2.7). Next, define X and $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ as follows. Let $X=\mathbb{R}$. Finally, define $\Phi:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ by

$$\Phi(\mathbf{i})=\log p_{\mathrm{i}(\mathbf{i})}$$

and let

$$U\mu=\frac{\int\Phi\,d\mu}{\int\Lambda\,d\mu}\,. \tag{4.7}$$

Note that if $\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}$, then $UL_n[\mathbf{i}]=\{\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\,|\,\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\}$. It therefore follows that if $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is a continuous function, then

$$\overline{P}_{C}^{U}(\varphi)=\limsup_{n}\frac{1}{n}\log\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ \forall\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\in C}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^{k}\mathbf{u}\right), \tag{4.8}$$

$$\overline{P}_{B(C,r)}^{U}(\varphi)=\limsup_{n}\frac{1}{n}\log\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ \forall\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\in B(C,r)}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^{k}\mathbf{u}\right), \tag{4.9}$$

and

$$\zeta_{C}^{\mathsf{dyn},U}(\varphi;z)=\sum_{n}\frac{z^{n}}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ \forall\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\in C}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^{k}\mathbf{u}\right). \tag{4.10}$$

For $\alpha\in\mathbb{R}$ and $C=\{\alpha\}$, we are now attempting to relate the solutions $\mathscr{f}(\alpha)$ and $\mathscr{F}(\alpha)$ of the following multifractal Bowen equations

$$\lim_{r\searrow 0}\overline{P}_{B(C,r)}^{U}(\mathscr{f}(\alpha)\,\Lambda)=0\,, \tag{4.11}$$

$$\overline{P}_{C}^{U}(\,\mathscr{F}(\alpha)\,\Lambda)=0\,, \tag{4.12}$$

to the multifractal spectrum $f_{\mu_{\mathsf{i}}}(\alpha)$ of μ_{i}; observe that the existence and uniqueness of the solutions $\mathscr{f}(\alpha)$ and $\mathscr{F}(\alpha)$ to (4.11) and (4.12) follow from Proposition 4.3. However, it is clear that if $C=\{\alpha\}$, then the sum in (4.8) may be empty, and the solution $\mathscr{F}(\alpha)$ to the equation $\overline{P}_{C}^{U}(\,\mathscr{F}(\alpha)\,\Lambda)=0$ is therefore $-\infty$, i.e. $\mathscr{F}(\alpha)=-\infty$. Hence, it may happen that

$$\mathscr{F}(\alpha)=-\infty<f_{\mu_{\mathsf{i}}}(\alpha)\,. \tag{4.13}$$

It follows from this discussion that if $C=\{\alpha\}$, then the pressure (4.8) and the zeta-function (4.10) do not, in general, encode sufficient information allowing us to recover the multifractal spectrum $f_{\mu_{\mathsf{i}}}(\alpha)$. The reason for the strict inequality in (4.13) is, of course, clear: even though there are no strings $\mathbf{i}\in\Sigma_{\mathsf{G}}^{*}$ and $\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})$ for which the ratio $\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}$ equals α, there may nevertheless be many sequences $(\mathbf{i}_n)_n$ of strings $\mathbf{i}_n\in\Sigma^{*}$ for which the sequence of sets $\{\frac{\log p_{\mathbf{i}_n}}{\log|DS_{\mathbf{i}_n}(\pi\mathbf{u})|}\,|\,\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i}_n)=\mathrm{i}(\mathbf{u})\}$ "shrinks" to the singleton $\{\alpha\}$. In order to capture this, we can proceed in two equally natural ways. Either, we can consider a family of enlarged "target" sets shrinking to the original main "target" $\{\alpha\}$; this approach will be referred to as the shrinking target approach. Or, alternatively, we can consider a fixed enlarge "target" set and regard this as our original main "target"; this approach will be referred to as the fixed target approach. Indeed, the statement of our main results below is divided into two parts paralleling the above discussion, namely: the first part (consisting of Theorem 4.4 and Corollary 4.5) presents our results in the shrinking target setting, and the second part consisting of Theorem 4.6 and Corollary 4.7) presents our results in the fixed target setting.

Statement of main results in the shrinking target setting. In the shrinking target setting, Theorem 4.4 provide a variational principle for the multifractal pressure and Corollary 4.5 provide a variational principle for the solution $f(C)$ to the multifractal Bowen equation (4.5). Below we denote the entropy of a measure $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by $h(\mu)$.

Theorem 4.4. The shrinking target variational principle for the multifractal pressure. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of X. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$.*

(1) *We have*

$$\lim_{r \searrow 0} \underline{P}^{U}_{B(C,r)}(\varphi) = \lim_{r \searrow 0} \overline{P}^{U}_{B(C,r)}(\varphi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} \left(h(\mu) + \int \varphi \, d\mu \right).$$

(2) *We have*

$$\lim_{r \searrow 0} -\log \sigma_{\mathsf{rad}}\big(\zeta^{\mathsf{dyn},U}_{B(C,r)}(\varphi;\cdot) \big) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} \left(h(\mu) + \int \varphi \, d\mu \right).$$

Theorem 4.4 is proved in Section 9 using techniques from large deviation theory developed in Sections 6–8. Observe that if we let $C = X$ in Theorem 4.4, then the multifractal pressure equals the usual pressure, i.e. $\overline{P}^{U}_{B(C,r)}(\varphi) = \underline{P}^{U}_{B(C,r)}(\varphi) = P(\varphi)$, and the variational principle in Theorem 4.4.(1) therefore simplifies to the usual variational principle, namely,

$$P(\varphi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \int \varphi \, d\mu \right).$$

Corollary 4.5. The shrinking target multifractal Bowen equation. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of X. Fix a continuous function $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ with $\Phi < 0$ and let $f(C)$ be the unique real number such that*

$$\lim_{r \searrow 0} \overline{P}^{U}_{B(C,r)}(\, f(C)\, \Phi) = 0\,;$$

alternatively, $f(C)$ is the unique real number such that

$$\lim_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\zeta^{U}_{B(C,r)}(f(C)\, \Phi;\cdot) \big) = 1\,.$$

Then

$$f(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} -\frac{h(\mu)}{\int \Phi \, d\mu}\,.$$

Proof
It follows from Theorem 4.4 and the definition of $f(C)$ that

$$\sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \overline{C}}} \left(h(\mu) + f(C) \int \Phi \, d\mu \right) = \lim_{r \searrow 0} \overline{P}^{U}_{B(C,r)}(\, f(C)\, \Phi) = 0\,. \tag{4.14}$$

The desired formula for $f(C)$ follows easily from (4.14). $\square$

Statement of main results in the fixed target setting. Of course, if the set C is "too small", then it follows from the discussion following the statement of Proposition 4.3 that we, in general, cannot expect any meaningful results in the fixed target setting. However, if the set C satisfies a non-degeneracy condition guaranteeing that it is not "too small" (namely condition (4.15) below), then meaningful results can be obtained in the fixed target setting. This is the contents of Theorem 4.6 and Corollary 4.7 below. Indeed, Theorem 4.6 and Corollary 4.7 provide variational principles for the multifractal pressure and for the solution $f(C)$ to the multifractal Bowen equation (4.6) in the fixed target setting.

Theorem 4.6. The fixed target variational principle for the multifractal pressure. *Let X be a normed vector space. Let $\Gamma : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous and affine and let $\Delta : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Define $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ by $U = \frac{\Gamma}{\Delta}$. Let C be a closed and convex subset of X and assume that*

$$\mathring{C} \cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big) \neq \varnothing\,. \tag{4.15}$$

(1) *We have*

$$P_C^U(\varphi) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in C}} \left(h(\mu) + \int \varphi \, d\mu \right) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \mathring{C}}} \left(h(\mu) + \int \varphi \, d\mu \right).$$

(2) *We have*

$$-\log \sigma_{\mathsf{rad}}\big(\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)\big) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in C}} \left(h(\mu) + \int \varphi \, d\mu \right) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in \mathring{C}}} \left(h(\mu) + \int \varphi \, d\mu \right).$$

Theorem 4.6 is proved in Section 10 using techniques from large deviation theory developed in Sections 6–8. Again, we observe that if we let $C = X$ in Theorem 4.6, then the multifractal pressure equals the usual pressure, i.e. $P_C^U(\varphi) = P(\varphi)$, and the variational principle in Theorem 4.6.(1) therefore simplifies to the usual variational principle, namely,

$$P(\varphi) = \sup_{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \left(h(\mu) + \int \varphi \, d\mu \right).$$

Corollary 4.7. The fixed target multifractal Bowen equation. *Let X be a normed vector space. Let $\Gamma : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous and affine and let $\Delta : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Define $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ by $U = \frac{\Gamma}{\Delta}$. Let C be a closed and convex subset of X and assume that*

$$\mathring{C} \cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big) \neq \varnothing\,.$$

Let $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be continuous with $\Phi < 0$. Let $f(C)$ be the unique real number such that

$$P_C^U\big(f(C)\,\Phi\big) = 0\,;$$

alternatively, $f(C)$ is the unique real number such that

$$\sigma_{\mathsf{rad}}\big(\zeta_C^U\big(f(C)\,\Phi;\cdot\big)\big) = 1\,.$$

Then

$$f(C) = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ U\mu \in C}} -\frac{h(\mu)}{\int \Phi \, d\mu}\,.$$

Proof
The proof is similar to the proof of Corollary 4.5 using Theorem 4.6 and the definition of $f(C)$. $\square$

In the next section we will apply Theorem 4.4, Corollary 4.5, Theorem 4.6 and Corollary 4.7 to show that in many cases, the solutions $f(C)$ and $f(C)$ to the multifractal Bowen equations (4.5) and (4.6) coincide with various well-known multifractal spectra.

Outline of the proofs of Theorem 4.4 and Theorem 4.6. Below we give an outline illustrating the key ideas in the proofs of the two main results, namely, Theorem 4.4 and Theorem 4.6. We first note that Theorem 4.6 follows from Theorem 4.4 by a "continuity" argument; see Section 10 for the details of this argument. We will now give an outline of the proof of Theorem 4.4. Recall the statement of Theorem 4.4. Namely, let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Also, let $C\subseteq X$ be a subset of X. Theorem 4.4 now says that if $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is a continuous function, then

$$\lim_{r\searrow 0}\underline{P}^{U}_{B(C,r)}(\varphi)=\lim_{r\searrow 0}\overline{P}^{U}_{B(C,r)}(\varphi)=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)\tag{4.16}$$

and

$$\lim_{r\searrow 0}-\log\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn},U}_{B(C,r)}(\varphi;\cdot)\,\big)=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right).\tag{4.17}$$

It is clear that (4.17) follows from (4.16) (see Proposition 4.2), and it therefore suffices to prove (4.16). Writing

$$D(C)=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right),$$

then (4.16) can be written as

$$\begin{aligned}\lim_{r\searrow 0}\liminf_{n}\ \frac{1}{n}\ \log\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ UL_n[\mathbf{i}]\subseteq B(C,r)}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}=D(C)\,,\\ \lim_{r\searrow 0}\limsup_{n}\ \frac{1}{n}\ \log\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ UL_n[\mathbf{i}]\subseteq B(C,r)}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\ \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}=D(C)\,.\end{aligned}\tag{4.18}$$

We will use techniques from large deviation theory in order to analyse the asymptotic behaviour of the sums on the left hand side of (4.18). In particular, we will use one of the most celebrated results from large deviation theory, namely, Varadhan's integral lemma. This result says that if X is a complete separable metric space and $(P_n)_n$ is a sequence of probability measures on X satisfying the large deviation property with rate constants $a_n\in(0,\infty)$ for $n\in\mathbb{N}$ and rate function $I:\mathbb{R}\to[-\infty,\infty]$ (this terminology will be explained in Section 7), then any bounded continuous function $F:X\to\mathbb{R}$ satisfies the following:

$$\lim_{n}\frac{1}{a_n}\log\int\exp(a_nF)\,dP_n=-\inf_{x\in X}(\,I(x)-F(x))\tag{4.19}$$

and if we define the probability measure Q_n on X by

$$Q_n(E)=\frac{\int_E\exp(a_nF)\,dP_n}{\int\exp(a_nF)\,dP_n}\quad\text{for Borel subsets } E \text{ of } X,\tag{4.20}$$

then the sequence $(Q_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function $(I-F)-\inf_{x\in X,I(x)<\infty}(I(x)-F(x))$.

In order to use the above results to analyse the asymptotic behaviour of the sums on the left hand side of (4.18) we must construct a continuous function $\widetilde{F}_\varphi : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ and a sequence of measures $\widetilde{\Pi}_n$ having the large deviation property with constants $(n)_n$ such that the sum

$$\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ UL_n[\mathbf{i}]\subseteq B(C,r)}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \tag{4.21}$$

can be expressed in terms of the integral

$$\int \exp(n\widetilde{F}_\varphi)\, d\widetilde{\Pi}_n$$

and the measure $\widetilde{Q}_{\varphi,n}$ defined by

$$\widetilde{Q}_{\varphi,n}(E) = \frac{\int_E \exp(n\widetilde{F}_\varphi)\, d\widetilde{\Pi}_n}{\int \exp(n\widetilde{F}_\varphi)\, d\widetilde{\Pi}_n} \quad \text{for Borel subsets } E \text{ of } \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}).$$

In fact, in stead of working with the sum in (4.21), we will (for technical reasons that are explaining in Section 6) work with a slightly modified sum, namely the sum where the function L_n has been replaced by a slightly different function M_n. We will thus construct a continuous function $F_\varphi : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ and a sequence of measures Π_n having the large deviation property with constants $(n)_n$ such that the sum

$$\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{i}]\subseteq B(C,r)}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \tag{4.22}$$

can be expressed in terms of the integral

$$\int \exp(nF_\varphi)\, d\Pi_n$$

and the measure $Q_{\varphi,n}$ defined by

$$Q_{\varphi,n}(E) = \frac{\int_E \exp(nF_\varphi)\, d\Pi_n}{\int \exp(nF_\varphi)\, d\Pi_n} \quad \text{for Borel subsets } E \text{ of } \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}).$$

Finally, we will obtain the asymptotic behaviour of the original sum in (4.21) from the the asymptotic behaviour of the modified sum in (4.22). The overall implementation of this strategy is divided into 4 steps described below.

Step 1 (Section 6): The map M_n.
We define the map $M_n : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and prove a number of "continuity" results showing that the maps M_n and L_n are "close" together.

Step 2 (Section 7): The measures Π_n.
Using the map M_n, we define the measures Π_n and prove (using a result from Orey & Pelikan [OrPe2]) that the sequence $(\Pi_n)_n$ has the large deviation property with constants $(n)_n$ and rate function I given by

$$I(\mu) = \begin{cases} \log\lambda - h(\mu) & \text{for } \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}); \\ \infty & \text{for } \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}). \end{cases}$$

where λ is a constant (in fact, λ is the entropy of the Parry measure on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$).

Step 3 (Section 8): The asymptotic behaviour of the modified sum (4.22).

Define the function $F_\varphi : \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ by $F_\varphi(\mu) = \int \varphi \, d\mu$. Let the measure Π_n be as in Step 2 and define the probability measure $Q_{\varphi,n}$ on $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by

$$Q_{\varphi,n}(E) = \frac{\int_E \exp(nF_\varphi) \, d\Pi_n}{\int \exp(nF_\varphi) \, d\Pi_n} \quad \text{for Borel subsets } E \text{ of } \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}).$$

We first show that the modified sum (4.22) can be expressed in terms of the integral $\int \exp(nF_\varphi) \, d\Pi_n$ and the measure $Q_{\varphi,n}$; more precisely, we show that there is a positive constant c such that

$$\sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} \sup_{\mathbf{u} \in [\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \le c \, \lambda^n \, Q_{\varphi,n}\Big(\{U \in C\}\Big) \int \exp(nF_\varphi) \, d\Pi_n \,,$$

$$\sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} \sup_{\mathbf{u} \in [\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \ge \frac{1}{c} \, \lambda^n \, Q_{\varphi,n}\Big(\{U \in C\}\Big) \int \exp(nF_\varphi) \, d\Pi_n \,,$$

for all n. Using Varadhan's integral lemma, this is easily seen to imply that if G is an open subset of X with $U^{-1}G \cap \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \neq \varnothing$, then

$$\liminf_n \frac{1}{n} \log \sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq G}} \sup_{\mathbf{u} \in [\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \ge D(G) \,, \tag{4.23}$$

and if K is a closed subset of X with $U^{-1}K \cap \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \neq \varnothing$, then

$$\limsup_n \frac{1}{n} \log \sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq K}} \sup_{\mathbf{u} \in [\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \le D(K) \,. \tag{4.24}$$

Step 4 (Section 9): The asymptotic behaviour of the (non-modified) sum (4.21).

Finally, using the "continuity" results from Step 1 (showing that the maps M_n and L_n are "close" together), the desired asymptotic behaviour, i.e. (4.18), of the (non-modified) sum (4.21) can be obtained from the asymptotic behaviour, i.e. (4.23) and (4.24), of the modified sum (4.22) established in Step 3.

5. Applications: multifractal spectra of measures and multifractal spectra of ergodic Birkhoff averages

We will now consider several of applications of Theorem 4.4, Corollary 4.5, Theorem 4.6 and Corollary 4.7 to multifractal spectra of measures and ergodic averages. In particular, we consider the following examples:

- Section 5.1: Multifractal spectra of graph-directed self-conformal measures.
- Section 5.2: Multifractal spectra of ergodic Birkhoff averages of continuous functions on graph-directed self-conformal sets.

5.1. Multifractal spectra of graph directed self-conformal measures. Let $(\mathsf{V}, \mathsf{E}, (V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}, (X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}, (S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}, (p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}})$ be a graph-directed conformal iterated function system with probabilities (see Section 2.2) and let $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ and $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the list of graph-directed self-conformal sets and the list of graph-direted self-conformal measures associated with the list $(\mathsf{V}, \mathsf{E}, (V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}, (X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}, (S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}, (p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}})$, respectively, i.e. the sets in the list $(K_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ are the unique non-empty compact sets satisfying (2.6) and the measures in the list $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ are the unique probability measures satisfying (2.7). Recall that the Hausdorff multifractal spectrum $f_{\mu_{\mathsf{i}}}$ of μ_{i} is defined by

$$f_{\mu_{\mathsf{i}}}(\alpha) = \dim_{\mathsf{H}}\left\{x \in K_{\mathsf{i}} \,\middle|\, \lim_{r\searrow 0}\frac{\log \mu_{\mathsf{i}}(B(x,r))}{\log r} = \alpha\right\},$$

for $\alpha \in \mathbb{R}$, see Section 1. If the OSC is satisfied, then the multifractal spectrum $f_{\mu_{\mathsf{i}}}(\alpha)$ can be computed as follows. Define $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ by

$$\Phi(\mathbf{i}) = \log p_{\mathsf{i}(\mathbf{i})}$$

for $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2\ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ and recall that the map $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$\Lambda(\mathbf{i}) = \log |DS_{\mathsf{e}_1}(\pi S\mathbf{i})|$$

for $\mathbf{i} = \mathsf{e}_1\mathsf{e}_2\ldots \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$. Next, we define the function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$P\big(\,\beta(q)\Lambda + q\Phi\,\big) = 0\,; \tag{5.1}$$

alternatively, the function $\beta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn}}(q\Phi + \beta(q)\Lambda;\cdot)\,\big) = 1\,. \tag{5.2}$$

We note that if the all maps S_{e} are similarities, i.e. if for each $\mathsf{e} \in \mathsf{E}$ there is a number $r_{\mathsf{e}} \in (0,1)$ such that

$$|S_{\mathsf{e}}(x) - S_{\mathsf{e}}(y)| = r_{\mathsf{e}}|x-y|$$

for all $x, y \in X_{\mathsf{t(e)}}$, then there is an alternative characterisation of the function β. Namely, in this case the function β is given by the following. For $q, t \in \mathbb{R}$, define the matrix $A(q,t) = \big(\,a_{\mathsf{i},\mathsf{j}}(q,t)\,\big)_{\mathsf{i},\mathsf{j}\in\mathsf{V}}$ by

$$a_{\mathsf{i},\mathsf{j}}(q,t) = \sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{i},\mathsf{j}}} p_{\mathsf{e}}^q r_{\mathsf{e}}^t\,.$$

For $q \in \mathbb{R}$, the number $\beta(q)$ is now the unique real number such that

$$\rho_{\mathsf{spec\text{-}rad}}\, A(q, \beta(q)) = 1\,;$$

here and below we use the following notation, namely, if M is a square matrix, then we will write $\rho_{\mathsf{spec\text{-}rad}}\, M$ for the spectral radius of M. If the OSC is satisfied, then the multifractal spectrum $f_{\mu_{\mathsf{i}}}$ of μ_{i} can be computer as follows, see Theorem D below. This result was first established by Edgar & Mauldin [EdMa] in 1992 assuming that the maps S_{e} were similarities and was subsequently extended to the conformal case by Cole [Col1,Col2] building on earlier results due to Arbeiter & Patzschke [ArPa] and Patzschke [Pa]. Below we use the following notation, namely, if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function, then $\varphi^* : \mathbb{R} \to [\infty, \infty]$ denotes the Legendre transform of φ defined by

$$\varphi^*(x) = \inf_y (xy + \varphi(y))\,.$$

We can now state Theorem D.

Theorem D [Col1,Pa]. *Let $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the list of graph-directed self-conformal measures defined by (2.7). Let $\alpha\in\mathbb{R}$. If the OSC is satisfied, then we have*

$$f_{\mu_{\mathsf{i}}}(\alpha)=\beta^*(\alpha)$$

for all $\mathsf{i}\in\mathsf{V}$.

As a first application of Theorem 4.4, Corollary 4.5, Theorem 4.6 and Corollary 4.7 we obtain a dynamical multifractal zeta-function with an associated Bowen equation whose solution equals the multifractal spectrum $f_{\mu_{\mathsf{i}}}(\alpha)$ of the graph-directed self-conformal measure μ_{i}. This is the content of the next theorem.

Theorem 5.1. Dynamical multifractal zeta-functinons for multifractal spectra of graph-directed self-conformal measures. *Let $(\mu_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ be the list of graph-directed self-conformal measures associated with the list $\big(\mathsf{V},\,\mathsf{E},\,(V_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\,(X_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}},\,(S_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}},\,(p_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}\big)$, i.e. μ_{i} is the unique probability measure such that $\mu_{\mathsf{i}}=\sum_{\mathsf{e}\in E_{\mathsf{i}}}p_{\mathsf{e}}\,\mu_{\mathsf{t}(\mathsf{e})}\circ S_{\mathsf{e}}^{-1}$.*

For $C\subseteq\mathbb{R}$ and a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we define the dynamical graph-directed self-conformal multifractal zeta-function by

$$\zeta_C^{\mathsf{dyn\text{-}con}}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\in C}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right)$$

for those complex numbers z for which the series converges. Let Λ be defined by (2.10) and let β be defined by (5.1) (or, alternatively, by (5.2)). Let $\alpha\in\mathbb{R}$.

(1) *There is a unique real number $\not{f}(\alpha)$ such that*
$$\lim_{r\searrow0}\sigma_{\mathsf{rad}}\big(\zeta_{B(\alpha,r)}^{\mathsf{dyn\text{-}con}}(\not{f}(\alpha)\,\Lambda;\cdot)\big)=1\,.$$
(2) *We have*
$$\not{f}(\alpha)=\beta^*(\alpha)\,.$$
(3) *If the OSC is satisfied, then we have*
$$\not{f}(\alpha)=f_{\mu_{\mathsf{i}}}(\alpha)=\dim_{\mathsf{H}}\left\{x\in K_{\mathsf{i}}\,\middle|\,\lim_{r\searrow0}\frac{\log\mu_{\mathsf{i}}(B(x,r))}{\log r}=\alpha\right\}$$
for all $\mathsf{i}\in\mathsf{V}$.

We will now prove Theorem 5.1. Recall that the function $\Lambda:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is defined by $\Lambda(\mathbf{i})=\log|DS_{\mathsf{e}_1}(\pi S\mathbf{i})|$ for $\mathbf{i}=\mathsf{e}_1\mathsf{e}_2\ldots\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$. Also, recall that $\Phi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ is defined by $\Phi(\mathbf{i})=\log p_{\mathsf{i}(\mathbf{i})}$ for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$. We now introduce the following definition. Define $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ by

$$U\mu=\frac{\int\Phi\,d\mu}{\int\Lambda\,d\mu}\,,\tag{5.3}$$

and note that if $\mathbf{i}\in\Sigma_{\mathsf{G}}^n$, then

$$UL_n[\mathbf{i}]=\left\{\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\,\middle|\,\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\right\}.$$

It therefore follows that

$$\zeta_C^{\mathsf{dyn},U}(\varphi;z)=\sum_n\frac{z^n}{n}\left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \forall\mathbf{u}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\text{ with }\mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\,\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\in C}}\ \sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\right).\tag{5.4}$$

In order to prove Theorem 5.1 we first prove three small auxiliary results, namely, Proposition 5.2, Proposition 5.3 and Proposition 5.4.

Proposition 5.2. *Let U be defined by (5.3). Let Λ be defined by (2.10) and let β be defined by (5.1) (or, alternatively, by (5.2)). Let $\alpha \in \mathbb{R}$. Then*

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int\Lambda\,d\mu} = \beta^*(\alpha)\,.$$

Proof
This result in folk-lore. However, for the sake of completeness we have decided to include the brief proof. We must the following two inequalities, namely

$$\beta^*(\alpha) \le \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int\Lambda\,d\mu}\,, \tag{5.5}$$

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int\Lambda\,d\mu} \le \beta^*(\alpha)\,. \tag{5.6}$$

Proof of (5.5). For $s \in \mathbb{R}$ and $q \in \mathbb{R}$, let $\mu_{s,q}$ denote the Gibbs state of $s\Lambda + q\Phi$. We now prove the following two claims.

Claim 1. For all q, we have $\frac{\int\Phi\,d\mu_{\beta(q),q}}{\int\Lambda\,d\mu_{\beta(q),q}} = -\beta'(q)$.
Proof of Claim 1. Define $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by $F(s,q) = P(\,s\Lambda + q\Phi\,)$ for $s, q \in \mathbb{R}$. It follows from [Rue1] that F is real analytic with $\frac{\partial}{\partial s}F(s,q) = \int\Lambda\,d\mu_{s,q}$ and $\frac{\partial}{\partial q}F(s,q) = \int\Phi\,d\mu_{s,q}$. Next, since $0 = F(\beta(q), q)$ for all q, it therefore follows from an application of the chain rule that

$$0 = \int\Lambda\,d\mu_{\beta(q),q}\;\beta'(q) \;+\; \int\Phi\,d\mu_{\beta(q),q}$$

for all q, whence $-\beta'(q) = \frac{\int\Phi\,d\mu_{\beta(q),q}}{\int\Lambda\,d\mu_{\beta(q),q}}$ for all q. This completes the proof of Claim 1.

Claim 2. For all q, we have $-\frac{h(\mu_{\beta(q),q})}{\int\Lambda\,d\mu_{\beta(q),q}} \ge \beta^*(-\beta'(q))$.
Proof of Claim 2. Since $\mu_{\beta(q),q}$ is a Gibbs state of $\beta(q)\Lambda + q\Phi$ and $P(\beta(q)\Lambda + q\Phi) = 0$, we deduce that

$$\begin{aligned}
0 &= P(\beta(q)\Lambda + q\Phi)\\
&= h(\mu_{\beta(q),q}) + \int(\beta(q)\Lambda + q\Phi)\,d\mu_{\beta(q),q}\\
&= h(\mu_{\beta(q),q}) + \beta(q)\int\Lambda\,d\mu_{\beta(q),q} + q\int\Phi\,d\mu_{\beta(q),q}\,,
\end{aligned}$$

whence $-\frac{h(\mu_{\beta(q),q})}{\int\Lambda\,d\mu_{\beta(q),q}} = \beta(q) + q\frac{\int\Phi\,d\mu_{\beta(q),q}}{\int\Lambda\,d\mu_{\beta(q),q}}$. It follows from this and Claim 1 that

$$-\frac{h(\mu_{\beta(q),q})}{\int\Lambda\,d\mu_{\beta(q),q}} = \beta(q) + q\frac{\int\Phi\,d\mu_{\beta(q),q}}{\int\Lambda\,d\mu_{\beta(q),q}} = \beta(q) - q\beta'(q) \ge \inf_p(\beta(p) - p\beta'(q)) = \beta^*(-\beta'(q))$$

for all q. This completes the proof of Claim 2.

We can now prove (5.5). If $\beta^*\alpha) = -\infty$, then inequality (5.5) is clear. Hence, we may assume that $\beta^*(\alpha) > -\infty$. In this case it follows from the convexity of β that there is a point $q_\alpha \in \mathbb{R}$ such that $\alpha = -\beta'(q_\alpha)$, see [Ro]. We therefore conclude from Claim 1 that the measure $\mu_{\beta(q_\alpha),q_\alpha}$ satisfies

$U\mu_{\beta(q_\alpha),q_\alpha} = \frac{\int \Phi\, d\mu_{\beta(q_\alpha),q_\alpha}}{\int \Lambda\, d\mu_{\beta(q_\alpha),q_\alpha}} = -\beta'(q_\alpha) = \alpha$, whence, using Claim 2, $\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,,\,U\mu=\alpha} -\frac{h(\mu)}{\int \Lambda\, d\mu} \geq -\frac{h(\mu_{\beta(q_\alpha),q_\alpha})}{\int \Lambda\, d\mu_{\beta(q_\alpha),q_\alpha}} \geq \beta^*(-\beta'(q_\alpha)) = \beta^*(\alpha)$. This completes the proof of (5.5).

Proof of (5.6). Fix $q\in\mathbb{R}$ and $\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with $U\mu=\alpha$. Using the variational principle (see [Wa]) we conclude that

$$
\begin{aligned}
P(\beta(q)\Lambda+q\Phi) &= \sup_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})} \Big(\, h(\nu) + \int (\beta(q)\Lambda+q\Phi)\, d\nu \,\Big)\\
&\geq h(\mu) + \int (\beta(q)\Lambda+q\Phi)\, d\mu\\
&= h(\mu) + \beta(q)\int \Lambda\, d\mu + q\int \Phi\, d\mu\,.
\end{aligned}
$$

Since $\Lambda<0$, this implies that $\beta(q)\geq -\frac{h(\mu)}{\int\Lambda\, d\mu} - q\frac{\int\Phi\, d\mu}{\int\Lambda\, d\mu}$. Next, as $\frac{\int\Phi\, d\mu}{\int\Lambda\, d\mu} = U\mu = \alpha$, we therefore conclude that

$$
\begin{aligned}
q\alpha+\beta(q) &\geq q\alpha - \frac{h(\mu)}{\int\Lambda\, d\mu} - q\frac{\int\Phi\, d\mu}{\int\Lambda\, d\mu}\\
&\geq q\alpha + \frac{h(\mu)}{\int\Lambda\, d\mu} - q\alpha\\
&= -\frac{h(\mu)}{\int\Lambda\, d\mu}\,.
\end{aligned}
$$

Finally, taking infimum over all $q\in\mathbb{R}$ and taking supremum over all $\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with $U\mu=\alpha$ gives inequality (5.6). □

Proposition 5.3. *Let U be defined by (5.3). Let Λ be defined by (2.10) and let β be defined by (5.1) (or, alternatively, by (5.2)). Let $\alpha\in\mathbb{R}$. Let t be the unique real number such that*

$$\lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn},U}_{B(\alpha,r)}(t\Lambda;\cdot)\,\big) = 1\,.$$

Then

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} \left(h(\mu) + t\int\Lambda\, d\mu\right) = 0\,,$$

and we have

$$t = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int\Lambda\, d\mu} = \beta^*(\alpha)\,.$$

Proof

We first note that it follows immediately from Theorem 4.4 that

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} \left(h(\mu) + t\int\Lambda\, d\mu\right) = \lim_{r\searrow 0}\overline{P}^{U}_{B(\alpha,r)}(t\Lambda) = \lim_{r\searrow 0} -\log\sigma_{\mathsf{rad}}\big(\,\zeta^{\mathsf{dyn},U}_{B(\alpha,r)}(t\Lambda;\cdot)\,\big) = 0\,. \tag{5.7}$$

Equality (5.7) is easily seen to imply that

$$t = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int\Lambda\, d\mu}\,.$$

Finally, it follows from the above equality and Proposition 5.2 that $t=\beta^*(\alpha)$. This completes the proof. □

Proposition 5.4. *Let U be defined by (5.3). Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$.*

(1) *There is a sequence $(\Delta_n)_n$ with $\Delta_n > 0$ and $\Delta_n \to 0$ such that for all closed subsets W of $\mathbb{R}$ and for all $n \in \mathbb{N}$, $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ and $\mathbf{u} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{i}) = \mathrm{i}(\mathbf{u})$, we have*

$$\operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|}\,,\,W\right) \le \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\,,\,W\right) \quad +\ \Delta_n\,, \tag{5.8}$$

$$\operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_{\mathbf{i}}}\,,\,W\right) \le \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log |DS_{\mathbf{i}}(\pi\mathbf{u})|}\,,\,W\right) \ +\ \Delta_n\,. \tag{5.9}$$

(2) *Let W be a closed subset of $\mathbb{R}$. For all $r > 0$, we have*

$$\sigma_{\mathsf{rad}}\big(\,\zeta_{B(W,r)}^{\mathsf{dyn},U}(\varphi;\cdot)\,\big) \le \sigma_{\mathsf{rad}}\big(\,\zeta_{W}^{\mathsf{dyn\text{-}con}}(\varphi;\cdot)\,\big)\,, \tag{5.10}$$

$$\sigma_{\mathsf{rad}}\big(\,\zeta_{B(W,r)}^{\mathsf{dyn\text{-}con}}(\varphi;\cdot)\,\big) \le \sigma_{\mathsf{rad}}\big(\,\zeta_{W}^{\mathsf{dyn},U}(\varphi;\cdot)\,\big)\,. \tag{5.11}$$

(3) *Let C be a closed subset of $\mathbb{R}$. Then we have*

$$\lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn\text{-}con}}(\varphi;\cdot)\,\big) = \lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn},U}(\varphi;\cdot)\,\big)\,.$$

Proof
(1) It is well-known and follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa]) that there is a constant $c > 0$ such that for all integers n and all $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ and all $\mathbf{u}, \mathbf{v} \in [\mathbf{i}]$, we have $\frac{1}{c} \le \frac{|DS_{\mathbf{i}}(\pi S^n\mathbf{u})|}{\operatorname{diam} K_{\mathbf{i}}} \le c$ and $\frac{1}{c} \le \frac{|DS_{\mathbf{i}}(\pi S^n\mathbf{u})|}{|DS_{\mathbf{i}}(\pi S^n\mathbf{v})|} \le c$. It is not difficult to see that the desired result follows from this.
(2) Fix $r > 0$. Let $(\Delta_n)_n$ be the sequence from Part (1). Since $\Delta_n \to 0$, we can find a positive integer N_r such that if $n \ge N_r$, then $\Delta_n < r$. Consequently, using (5.9) in Part (1), for all $n \ge N_r$, we have

$$\begin{aligned}
\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ UL_n[\mathbf{i}]\subseteq W}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}
&= \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \forall\mathbf{u}\in\Sigma^{\mathbb{N}} \text{ with } \mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\, \frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|}\in W}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\\
&= \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \forall\mathbf{u}\in\Sigma^{\mathbb{N}} \text{ with } \mathrm{t}(\mathbf{i})=\mathrm{i}(\mathbf{u})\,:\, \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log|DS_{\mathbf{i}}(\pi\mathbf{u})|},W\right)=0}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\\
&\le \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},W\right)\le 0+\Delta_r}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\\
&\le \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \operatorname{dist}\left(\frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}},W\right)< r}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\\
&= \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \frac{\log p_{\mathbf{i}}}{\log\operatorname{diam}K_{\mathbf{i}}}\in B(W,r)}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp\sum_{k=0}^{n-1}\varphi S^k\mathbf{u}\,.
\end{aligned} \tag{5.12}$$

A similar argument using (5.8) in Part (1) shows that

$$\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ \frac{\log p_{\mathbf{i}}}{\log \operatorname{diam} K_i}\in W}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k\mathbf{u} \leq \sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ UL_n[\mathbf{i}]\subseteq B(W,r)}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k\mathbf{u}\,. \tag{5.13}$$

The desired results follow immediately from inequalities (5.12) and (5.13).
(3) This result follows easily from Part (2). $\square$

We can now prove Theorem 5.1.

Proof of Theorem 5.1
Let U be defined by (5.3).
(1) It follows from Proposition 5.4 that

$$\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn\text{-}con}}(t\Lambda;\cdot)\,\big) = \lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn},U}(t\Lambda;\cdot)\,\big) \tag{5.14}$$

for all t. Also, it follows from Proposition 5.2 that there is a unique number $\mathscr{f}(\alpha)$ such that

$$\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn},U}(\mathscr{f}(\alpha)\,\Lambda;\cdot)\,\big) = 1\,. \tag{5.15}$$

Combining (5.14) and (5.15) shows that $\mathscr{f}(\alpha)$ is the unique real number such we have the following $\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn\text{-}con}}(\mathscr{f}(\alpha)\,\Lambda;\cdot)\,\big) = \lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(\alpha,r)}^{\mathsf{dyn},U}(\mathscr{f}(\alpha)\,\Lambda;\cdot)\,\big) = 1$.
(2) It follows from (1) and Proposition 5.4 that

$$1 = \lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn\text{-}con}}(\mathscr{f}(\alpha)\,\Lambda;\cdot)\,\big) = \lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn},U}(\mathscr{f}(\alpha)\,\Lambda;\cdot)\,\big)\,,$$

and Corollary 4.5 therefore implies that the number $\mathscr{f}(\alpha)$ is given by

$$\mathscr{f}(\alpha) = \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu=\alpha}} -\frac{h(\mu)}{\int \Lambda\,d\mu}\,. \tag{5.16}$$

Finally, combining Proposition 5.2 and (5.16) shows that $\mathscr{f}(\alpha) = \beta^*(\alpha)$.
(3) This follows immediately from (2) and Theorem D. $\square$

5.2. Multifractal spectra of ergodic Birkhoff averages. We first fix $\gamma\in(0,1)$ and define the metric d_γ on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ as follows. For $\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathbf{i}\neq\mathbf{j}$, we will write $\mathbf{i}\wedge\mathbf{j}$ for the longest common prefix of $\mathbf{i}$ and $\mathbf{j}$ (i.e. $\mathbf{i}\wedge\mathbf{j}=\mathbf{u}$ where $\mathbf{u}$ is the unique element in Σ_{G}^* for which there are $\mathbf{k}',\mathbf{k}''\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathbf{k}'=\mathsf{e}_1'\mathsf{e}_2'\ldots$ and $\mathbf{k}''=\mathsf{e}_1''\mathsf{e}_2''\ldots$ such that $\mathsf{e}_1'\neq\mathsf{e}_1''$, $\mathbf{i}=\mathbf{u}\mathbf{k}'$ and $\mathbf{j}=\mathbf{u}\mathbf{k}''$). The metric d_γ is now defined by

$$\mathsf{d}_\gamma(\mathbf{i},\mathbf{j}) = \begin{cases} 0 & \text{if } \mathbf{i}=\mathbf{j};\\ \gamma^{|\mathbf{i}\wedge\mathbf{j}|} & \text{if } \mathbf{i}\neq\mathbf{j}, \end{cases}$$

for $\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$; throughout this section, we equip $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with the metric d_γ, and continuity and Lipschitz properties of functions $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ from $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ to $\mathbb{R}$ will always refer to the metric d_γ. Multifractal analysis of Birkhoff averages has received significant interest during the past 10 years, see, for example, [BaMe,FaFe,FaFeWu,FeLaWu,Oli,Ol1,OlWi]. Fix a positive integer M. The multifractal spectrum $F_{\mathbf{f}}^{\mathsf{erg}}$ of ergodic Birkhoff averages of a vector valued continuous function $\mathbf{f}:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}^M$ is defined by

$$F_{\mathbf{f}}^{\mathsf{erg}}(\boldsymbol{\alpha}) = \dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{f}(S^k\mathbf{i}) = \boldsymbol{\alpha}\right\}$$

for $\boldsymbol{\alpha} \in \mathbb{R}^M$; recall, that the map π is defined in Section 2. One of the main problems in multifractal analysis of Birkhoff averages is the detailed study of the multifractal spectrum $F_{\mathbf{f}}^{\mathsf{erg}}$. For example, Theorem E below is proved in different settings and at various levels of generality in [FaFe,FaFeWu,FeLaWu,Oli,Ol1,OlWi]. Before we can state this result we introduce the following notation. If $(x_n)_n$ is a sequence of points in a metric space X, then we write $\operatorname{acc}_n x_n$ for the set of accumulation points of the sequence $(x_n)_n$, i.e.

$$\operatorname*{acc}_n x_n = \left\{ x \in X \,\middle|\, x \text{ is an accumulation point of } (x_n)_n \right\}.$$

We will also use the following notation. Namely, if $\mathbf{f} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ is a continuous function with $\mathbf{f} = (f_1, \ldots, f_M)$, then we will write

$$\int \mathbf{f}\, d\mu = \left(\int f_1\, d\mu \,,\, \ldots \,,\, \int f_M\, d\mu \right)$$

for $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. We can now state Theorem E.

Theorem E [FaFe,FaFeWu,FeLaWu,Oli,Ol1,OlWi]. *Fix $\gamma \in (0,1)$ and let $\mathbf{f} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ be a Lipschitz function with respect to the metric d_γ. Let $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10). Let C be a closed subset of $\mathbb{R}^M$. If the OSC is satisfied, then*

$$\dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) \subseteq C \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \int \mathbf{f}\, d\mu \in C}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\,.$$

In particular, if the OSC is satisfied and $\boldsymbol{\alpha} \in \mathbb{R}^M$, then we have

$$\dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) = \boldsymbol{\alpha} \right\} = \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \int \mathbf{f}\, d\mu = \boldsymbol{\alpha}}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\,.$$

As a second application of Theorem 4.4, Corollary 4.5, Theorem 4.6 and Corollary 4.7 we will now obtain zeta-functions whose radii of convergence equal different types of multifractal spectra of ergodic Birkhoff averages. We first state and prove a rather general result, namely Theorem 5.5 below, from which analogous results for a number of different multifractal spectra of ergodic Birkhoff averages, including $F_{\mathbf{f}}^{\mathsf{erg}}(\boldsymbol{\alpha})$, can be deduced. Indeed, immediately after the statement and proof of Theorem 5.5, we will apply Theorem 5.5 to prove the following results, namely: Theorem 5.6 on the multifractal spectra of ergodic averages of continuous vector valued functions, Theorem 5.7 on the multifractal spectra of relative ergodic averages of continuous functions, and finally Theorem 5.8 on the multifractal spectra of a more general type of relative ergodic averages of continuous functions.

Theorem 5.5. Multifractal zeta-functions for abstract multifractal spectra of ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and $W \subseteq \mathbb{R}^I$ and let $\boldsymbol{\Phi} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^I$ be a Lipschitz function with respect to the metric d_γ such that $\{\int \boldsymbol{\Phi}\, d\mu \,|\, \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\} \subseteq W$; recall, that if $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_I)$, then we write $\int \boldsymbol{\Phi}\, d\mu = (\int \Phi_1\, d\mu, \ldots, \int \Phi_I\, d\mu)$. Let $Q : W \to \mathbb{R}^M$ be a continuous function.*

For $C \subseteq \mathbb{R}^M$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the abstract dynamical ergodic multifractal zeta-function associated with Q by

$$\zeta_C^{\mathsf{dyn\text{-}erg}}(\varphi; z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ \forall \mathbf{u} \in [\mathbf{i}] \,:\, Q\left(\frac{1}{n} \sum_{k=0}^{n-1} \boldsymbol{\Phi}(S^k \mathbf{u}) \right) \in C}} \sup_{\mathbf{u} \in [\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

Let $\Lambda:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$ be defined by (2.10).
(1) *Assume that $C\subseteq\mathbb{R}^M$ is closed.*
(1.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r\searrow 0}\sigma_{\mathsf{rad}}\big(\,\zeta_{B(C,r)}^{\mathsf{dyn\text{-}erg}}(\mathscr{f}(C)\,\Lambda;\cdot)\,\big)=1\,.$$

(1.2) *We have*

$$\mathscr{f}(C)=\sup_{\boldsymbol{\alpha}\in C}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ Q\left(\int\mathbf{\Phi}\,d\mu\right)=\boldsymbol{\alpha}}}-\frac{h(\mu)}{\int\Lambda\,d\mu}\,.$$

(1.3) *If the OSC is satisfied, then we have*

$$\mathscr{f}(C)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\operatorname*{acc}_n Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{i})\right)\subseteq C\right\}.$$

(2) *Assume that there are continuous and affine functions $\Gamma:W\to\mathbb{R}^M$ and $\Delta:W\to\mathbb{R}$ with $\Delta(\mathbf{x})\neq 0$ for all $\mathbf{x}$ such that $Q=\frac{\Gamma}{\Delta}$, and assume that $C\subseteq\mathbb{R}^M$ is closed and convex with $\mathring{C}\cap\{\frac{\Gamma(\int\mathbf{\Phi}\,d\mu)}{\Delta(\int\mathbf{\Phi}\,d\mu)}\,|\,\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\}\neq\varnothing$.*
(2.1) *There is a unique real number $\mathscr{F}(C)$ such that*

$$\sigma_{\mathsf{rad}}\big(\,\zeta_{C}^{\mathsf{dyn\text{-}erg}}(\mathscr{F}(C)\,\Lambda;\cdot)\,\big)=1\,.$$

(2.2) *We have*

$$\mathscr{F}(C)=\sup_{\boldsymbol{\alpha}\in C}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ Q\left(\int\mathbf{\Phi}\,d\mu\right)=\boldsymbol{\alpha}}}-\frac{h(\mu)}{\int\Lambda\,d\mu}\,.$$

(2.3) *If the OSC is satisfied, then we have*

$$\mathscr{F}(C)=\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\operatorname*{acc}_n Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{i})\right)\subseteq C\right\}.$$

Proof
We first note that it follows from [Ol1] that if C is a closed subset of $\mathbb{R}^M$, then

$$\dim_{\mathsf{H}}\pi\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,\operatorname*{acc}_n Q\left(\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{\Phi}(S^k\mathbf{i})\right)\subseteq C\right\}=\sup_{\boldsymbol{\alpha}\in C}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ Q\left(\int\mathbf{\Phi}\,d\mu\right)=\boldsymbol{\alpha}}}-\frac{h(\mu)}{\int\Lambda\,d\mu}\,. \tag{5.17}$$

Next, we define the function $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}^M$ to be the composite of the following 2 maps, namely,

$$\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\xrightarrow{\mu\to\int\mathbf{\Phi}\,d\mu}W\,,\quad W\xrightarrow{Q}\mathbb{R}^M\,,$$

i.e.

$$U\mu=Q\big(\textstyle\int\mathbf{\Phi}\,d\mu\,\big)$$

for $\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Since clearly $\zeta_C^{\mathsf{dyn\text{-}erg}}(\varphi;\cdot)=\zeta_C^{\mathsf{dyn},U}(\varphi;\cdot)$, the results now follow from Corollary 4.5, Corollary 4.7 and (5.17). $\square$

Next, we present three corollaries of Theorem 5.5.

Theorem 5.6. Multifractal zeta-functinons for multifractal spectra of vector valued ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and let $\mathbf{f} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ be a Lipschitz function with respect to the metric d_γ. For $C \subseteq \mathbb{R}^M$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the dynamical ergodic multifractal zeta-function by*

$$\zeta_C^{\mathsf{dyn\text{-}vec\text{-}erg}}(\varphi; z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ \forall \mathbf{u} \in [\mathbf{i}] \,:\, \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{u}) \in C}} \sup_{\mathbf{u} \in [\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

Let $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10).
(1) *Assume that $C \subseteq \mathbb{R}^M$ is closed.*
(1.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn\text{-}vec\text{-}erg}}(\mathscr{f}(C)\,\Lambda; \cdot) \big) = 1\,.$$

If $\boldsymbol{\alpha} \in \mathbb{R}^M$ and $C = \{\boldsymbol{\alpha}\}$, then we will write $\mathscr{f}(\boldsymbol{\alpha}) = \mathscr{f}(C)$.
(1.2) *We have*

$$\mathscr{f}(C) = \sup_{\boldsymbol{\alpha} \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \int \mathbf{f}\, d\mu = \boldsymbol{\alpha}}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\,.$$

(1.3) *If the OSC is satisfied, then we have*

$$\mathscr{f}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) \subseteq C \right\}.$$

In particular, if the OSC is satisfied and $\boldsymbol{\alpha} \in \mathbb{R}^M$, then we have

$$\mathscr{f}(\boldsymbol{\alpha}) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) = \boldsymbol{\alpha} \right\}.$$

(2) *Assume that $C \subseteq \mathbb{R}^M$ is closed and convex with $\mathring{C} \cap \{\int \mathbf{f}\, d\mu \,|\, \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\} \neq \varnothing$.*
(2.1) *There is a unique real number $\mathscr{F}(C)$ such that*

$$\sigma_{\mathsf{rad}}\big(\zeta_C^{\mathsf{dyn\text{-}vec\text{-}erg}}(\mathscr{F}(C)\,\Lambda; \cdot) \big) = 1\,.$$

(2.2) *We have*

$$\mathscr{F}(C) = \sup_{\boldsymbol{\alpha} \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \int \mathbf{f}\, d\mu = \boldsymbol{\alpha}}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\,.$$

(2.3) *If the OSC is satisfied, then we have*

$$\mathscr{F}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname*{acc}_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{f}(S^k \mathbf{i}) \subseteq C \right\}.$$

Proof
This follows immediately by applying Theorem 5.5 to $W = \mathbb{R}^M$ and the maps $\mathbf{\Phi} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^M$ and $Q : W \to \mathbb{R}^M$ defined by $\mathbf{\Phi} = \mathbf{f}$ and $Q(\mathbf{x}) = \mathbf{x}$. $\square$

Theorem 5.7. Multifractal zeta-functinons for multifractal spectra of relative ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and let $f, g : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be Lipschitz functions with respect to the metric d_γ and assume that $g(\mathbf{i}) \neq 0$ for all $\mathbf{i}$. For $C \subseteq \mathbb{R}$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the dynamical relative ergodic multifractal zeta-function by*

$$\zeta_C^{\mathsf{dyn\text{-}rel\text{-}erg}}(\varphi; z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ \forall \mathbf{u} \in [\mathbf{i}] \,:\, \frac{\sum_{k=0}^{n-1} f(S^k \mathbf{u})}{\sum_{k=0}^{n-1} g(S^k \mathbf{u})} \in C}} \sup_{\mathbf{u} \in [\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \right).$$

Let $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10).

(1) *Assume that $C \subseteq \mathbb{R}^M$ is closed.*

(1.1) *There is a unique real number $\mathscr{f}(C)$ such that*

$$\lim_{r \searrow 0} \sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\mathsf{dyn\text{-}rel\text{-}erg}}(\mathscr{f}(C)\,\Lambda; \cdot)\big) = 1\,.$$

If $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$, then we will write $\mathscr{f}(\alpha) = \mathscr{f}(C)$.

(1.2) *We have*

$$\mathscr{f}(C) = \sup_{\alpha \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \frac{\int f\,d\mu}{\int g\,d\mu} = \alpha}} -\frac{h(\mu)}{\int \Lambda\,d\mu}\,.$$

(1.3) *If the OSC is satisfied, then we have*

$$\mathscr{f}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname{acc}_n \frac{\sum_{k=0}^{n-1} f(S^k \mathbf{i})}{\sum_{k=0}^{n-1} g(S^k \mathbf{i})} \subseteq C \right\}.$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\mathscr{f}(\alpha) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \lim_n \frac{\sum_{k=0}^{n-1} f(S^k \mathbf{i})}{\sum_{k=0}^{n-1} g(S^k \mathbf{i})} = \alpha \right\}.$$

(2) *Assume that $C \subseteq \mathbb{R}^M$ is closed and convex with $\mathring{C} \cap \{\frac{\int f\,d\mu}{\int g\,d\mu} \,|\, \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\} \neq \varnothing$.*

(2.1) *There is a unique real number $\mathscr{F}(C)$ such that*

$$\sigma_{\mathsf{rad}}\big(\zeta_C^{\mathsf{dyn\text{-}rel\text{-}erg}}(\mathscr{F}(C)\,\Lambda; \cdot)\big) = 1\,.$$

(2.2) *We have*

$$\mathscr{F}(C) = \sup_{\alpha \in C} \sup_{\substack{\mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \frac{\int f\,d\mu}{\int g\,d\mu} = \alpha}} -\frac{h(\mu)}{\int \Lambda\,d\mu}\,.$$

(2.3) *If the OSC is satisfied, then we have*

$$\mathscr{F}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, \operatorname{acc}_n \frac{\sum_{k=0}^{n-1} f(S^k \mathbf{i})}{\sum_{k=0}^{n-1} g(S^k \mathbf{i})} \subseteq C \right\}.$$

Proof

This follows immediately by applying Theorem 5.5 to $W = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and the maps $\mathbf{\Phi} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^2$ and $Q : W \to \mathbb{R}$ defined by $\mathbf{\Phi} = (f, g)$ and $Q(x, y) = \frac{x}{y}$. $\square$

As a final application of Theorem 5.5 we consider a type of relative ergodic multifractal spectra involving quantities similar to those appearing in Hölder's inequality; for this reason we have decided to refer to these multifractal spectra as "Hölder-like relative ergodic Birkhoff averages".

Theorem 5.8. Multifractal zeta-functinons for multifractal spectra of Hölder-like relative ergodic Birkhoff averages. *Fix $\gamma \in (0,1)$ and let $f_1,\dots,f_M,g_1,\dots,g_M : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be Lipschitz functions with respect to the metric d_γ and assume that $f_l(\mathbf{i}) > 0$ for all l and all $\mathbf{i}$, and that $g_l(\mathbf{i}) > 0$ for all l and all $\mathbf{i}$. Fix $s_1,\dots,s_M,t_1,\dots,t_M > 0$. For $C \subseteq \mathbb{R}$ and a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define the dynamical Hölder-like relative ergodic multifractal zeta-function by*

$$\zeta_C^{\textsf{dyn-Höl-erg}}(\varphi; z) = \sum_n \frac{z^n}{n} \left(\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n \\ \forall \mathbf{u}\in[\mathbf{i}]\,:\, \frac{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_l(S^k\mathbf{u})\right)^{s_l}}{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_l(S^k\mathbf{u})\right)^{t_l}} \in C}} \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k\mathbf{u} \right).$$

Let $\Lambda : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be defined by (2.10). Assume that $C \subseteq \mathbb{R}^M$ is closed.

(1) *There is a unique real number $\not{f}(C)$ such that*

$$\lim_{r\searrow 0} \sigma_{\mathsf{rad}}\big(\zeta_{B(C,r)}^{\textsf{dyn-Höl-erg}}(\not{f}(C)\,\Lambda; \cdot)\big) = 1\,.$$

If $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$, then we will write $\not{f}(\alpha) = \not{f}(C)$.

(2) *We have*

$$\not{f}(C) = \sup_{\alpha\in C} \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \\ \frac{\prod_{l=1}^M \left(\int f_l\,d\mu\right)^{s_l}}{\prod_{l=1}^M \left(\int g_l\,d\mu\right)^{t_l}} = \alpha}} -\frac{h(\mu)}{\int \Lambda\, d\mu}\,.$$

(3) *If the OSC is satisfied, then we have*

$$\not{f}(C) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \;\middle|\; \operatorname*{acc}_n \frac{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_l(S^k\mathbf{u})\right)^{s_l}}{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_l(S^k\mathbf{u})\right)^{t_l}} \subseteq C \right\}.$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\not{f}(\alpha) = \dim_{\mathsf{H}} \pi \left\{ \mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \;\middle|\; \lim_n \frac{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} f_l(S^k\mathbf{u})\right)^{s_l}}{\prod_{l=1}^M \left(\frac{1}{n}\sum_{k=0}^{n-1} g_l(S^k\mathbf{u})\right)^{t_l}} = \alpha \right\}.$$

Proof
This follows immediately by applying Theorem 5.5 to $W = \mathbb{R}^M \times (\mathbb{R}\setminus\{0\})^M$ and the maps $\mathbf{\Phi} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}^{2M}$ and $Q : W \to \mathbb{R}$ defined by $\mathbf{\Phi} = (f_1,\dots,f_M,g_1,\dots,g_M)$ and $Q(x_1,\dots,x_M,y_1,\dots,y_M) = \frac{\prod_{l=1}^M x_l^{s_l}}{\prod_{l=1}^M y_l^{t_l}}$. $\square$

6. Proofs. The map M_n

The purpose of this section to introduce the auxiliary map M_n and to prove various continuity results regarding this map. The two main results are Lemma 6.2 and Lemma 6.3. Loosely speaking these lemmas say that the maps L_n and M_n behave asymptotically in the same way. We also state and prove a simple but useful lemma about upper semi-continuous maps, see Lemma 6.4. All the key lemmas is this section (i.e. Lemma 6.2, Lemma 6.3 and Lemma 6.4) play important roles in the following sections.

The map M_n. Since the graph $\mathsf{G}=(\mathsf{V},\mathsf{E})$ is strongly connected, it follows that for each $\mathbf{i}\in\Sigma_{\mathsf{G}}^{*}$, we can choose $\widehat{\mathbf{i}}\in\Sigma_{\mathsf{G}}^{*}$ with $|\,\widehat{\mathbf{i}}\,|\le|\mathsf{V}|$ such that $\mathrm{t}(\mathbf{i})=\mathrm{i}(\,\widehat{\mathbf{i}}\,)$ and $\mathrm{t}(\,\widehat{\mathbf{i}}\,)=\mathrm{i}(\mathbf{i})$. Next, for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{*}$, define $\overline{\mathbf{i}}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ by

$$\overline{\mathbf{i}}=\mathbf{i}\,\widehat{\mathbf{i}}\,\mathbf{i}\,\widehat{\mathbf{i}}\,\mathbf{i}\,\widehat{\mathbf{i}}\ldots$$

Finally, for a positive integer n, we define $M_n:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by

$$\begin{aligned}M_n\mathbf{i}&=L_{n+|\,\widehat{\mathbf{i}|n}\,|}\Big(\overline{\mathbf{i}|n}\Big)\\&=\frac{1}{n+|\,\widehat{\mathbf{i}|n}\,|}\sum_{k=0}^{n+|\,\widehat{\mathbf{i}|n}\,|-1}\delta_{S^k(\,\overline{\mathbf{i}|n}\,)}\end{aligned}\tag{6.1}$$

for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$; recall, that the map $L_n:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is defined by by

$$L_n\mathbf{i}=\frac{1}{n}\sum_{k=0}^{n-1}\delta_{S^k\mathbf{i}}$$

for $\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$; see (4.1).

Why the map M_n? We will now explain the main technical reason for introducing the map M_n. In order to prove Theorem 4.4 we will use results from large deviation theory. In particular, we will use Varadhan's integral lemma (i.e. Theorem 8.1) which says that if X is a complete separable metric space and $(P_n)_n$ is a sequence of probability measures on X satisfying the large deviation property with rate constants $a_n\in(0,\infty)$ for $n\in\mathbb{N}$ and rate function $I:\mathbb{R}\to[-\infty,\infty]$ (this terminology will be explained in Section 7), then

$$\lim_n\frac{1}{a_n}\log\int\exp(a_nF)\,dP_n=-\inf_{x\in X}(\,I(x)-F(x))$$

for any bounded continuous function $F:X\to\mathbb{R}$ (see Section 8 for more a detailed discussion and statement of this result).

More precisely, in Section 8 we use Varadhan's integral lemma to analyse the asymptotic behaviour of the integral

$$\frac{1}{n}\log\int\exp(nF_\varphi(L_n(\overline{\mathbf{i}|n})))\,d\Pi(\mathbf{i})\tag{6.2}$$

as $n\to\infty$ where Π is the Parry measure on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ (the Parry measure will be defined in Section 7) and the function $F_\varphi:\mathcal{P}\big(\Sigma_{\mathsf{G}}^{\mathbb{N}}\big)\to\mathbb{R}$ is given by $F_\varphi(\mu)=\int\varphi\,d\mu$. Defining $\Lambda_n:\mathcal{P}\big(\Sigma_{\mathsf{G}}^{\mathbb{N}}\big)\to\mathbb{R}$ by $\Lambda_n(\mathbf{i})=L_n(\,\overline{\mathbf{i}|n}\,)$, then (6.2) can be written as

$$\frac{1}{n}\log\int\exp(nF_\varphi)\,d(\Pi\circ\Lambda_n^{-1})\,.\tag{6.3}$$

Consequently, if the sequence $(\Pi\circ\Lambda_n^{-1})_n$ satisfied the large deviation property with rate constants $a_n=n$, then Varadhan's integral lemma could be applied to analyse the asymptotic behaviour of the sequence of integrals in (6.3). However, it follows from results by Orey & Pelikan [OrPe1,OrPe2] that the sequence $(\Pi\circ L_n)_n$ satisfies the large deviation property with rate constants $a_n=n$ and

Varadhan's integral lemma can therefore be applied to provide information about the asymptotic behaviour of the sequence of integral defined by

$$\frac{1}{n}\log\int\exp(nF_\varphi)\,d(\Pi\circ L_n^{-1})\,. \tag{6.4}$$

In order to utilise the knowledge of the asymptotic behaviour of (6.4) for analysing the asymptotic behaviour of (6.3), we must therefore show that the measures

$$\Pi\circ\Lambda_n^{-1}$$

and

$$\Pi\circ L_n^{-1}$$

are "close". However, for technical reasons we will prove and use a similar "closeness" statement involving the measures

$$\Pi\circ M_n^{-1}$$

and

$$\Pi\circ L_n^{-1}\,.$$

Indeed, below we prove a number of results showing that the maps M_n and L_n (and therefore also the measures $\Pi\circ M_n^{-1}$ and $\Pi\circ L_n^{-1}$) are "close". These results play an important role in Section 7. In particular, they allow us to: (1) use Orey & Pelikan's result from [OrPe1,OrPe2] saying that the sequence $(\Pi\circ L_n^{-1})_n$ satisfies the large deviation property to prove that the sequence $(\Pi\circ M_n^{-1})_n$ also satisfies the large deviation property (see Theorem 7.2), and (2) replace all occurencies of $L_n(\overline{\mathbf{i}|n})$ in the formulas in Section 8 by $M_n\mathbf{i}$ allowing us to use the large deviation property of the sequence $(\Pi\circ M_n^{-1})_n$. This explains the mean reason for introducing the map M_n and the associated measure $\Pi\circ M_n^{-1}$.

Comparing M_n and L_n. We now prove various continuity statements saying that the maps M_n and L_n are "close". These statements play an important role in Section 7 where we apply Varadhan's integral lemma to prove Theorem 7.2. We first introduce the metric L on $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Fix $\gamma\in(0,1)$ and let d_γ denote the metric on $\Sigma_{\mathsf{G}}^{\mathbb{N}}$ introduced in Section 5.2. For a function $f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$, we let $\mathrm{Lip}_\gamma(f)$ denote the Lipschitz constant of f with respect to the metric d_γ, i.e. $\mathrm{Lip}_\gamma(f)=\sup_{\mathbf{i},\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}},\mathbf{i}\neq\mathbf{j}}\frac{|f(\mathbf{i})-f(\mathbf{j})|}{\mathsf{d}_\gamma(\mathbf{i},\mathbf{j})}$ and we define the metric L in $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by

$$\mathsf{L}(\mu,\nu)=\sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\le1}}\left|\int f\,d\mu-\int f\,d\nu\right|; \tag{6.5}$$

we note that it is well-known that L is a metric and that L induces the weak topology. Below we will always equip the space $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with the metric L. Before stating and proving the key results in this section, we start by proving a small technical auxiliary result.

Lemma 6.1. *Let (X,d) be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix $r>0$. There is a positive integer N_r such that if $n\ge N_r$, $\mathbf{u}\in\Sigma_{\mathsf{G}}^{n}$ and $\mathbf{k},\mathbf{l}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u})=\mathrm{i}(\mathbf{l})$, then we have*

$$\mathsf{d}\big(UL_n(\mathbf{uk})\,,\,UM_n(\mathbf{ul})\big)<r\,.$$

Proof
Fix $\gamma\in(0,1)$ and let L be the metric in $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ defined in (6.5). Since $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ is continuous and $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is compact, we conclude that $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ is uniformly continuous. This implies that we can choose $\delta>0$ such that all measures $\mu,\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ satisfy the following implication:

$$\mathsf{L}(\mu,\nu)<\delta\quad\Rightarrow\quad\mathsf{d}(U\mu,U\nu)<\frac{r}{2}\,. \tag{6.6}$$

Next, choose a positive integer N_r such that $2\frac{|\mathsf{V}|}{N_r} < \delta$ and $\frac{1}{N_r(1-\gamma)} < \delta$.

Now, fix $n \geq N_r$, $\mathbf{u} \in \Sigma_{\mathsf{G}}^n$ and $\mathbf{k}, \mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$. It follows that

$$\begin{aligned}\mathsf{L}\big(\,L_n(\mathbf{uk})\,,\,M_n(\mathbf{ul})\,\big) &\leq \mathsf{L}\big(\,L_n(\mathbf{uk})\,,\,L_n(\mathbf{ul})\,\big) + \mathsf{L}\big(\,L_n(\mathbf{ul})\,,\,M_n(\mathbf{ul})\,\big)\\ &\leq \mathsf{L}\big(\,L_n(\mathbf{uk})\,,\,L_n(\mathbf{ul})\,\big) + \mathsf{L}\big(\,L_n(\mathbf{ul})\,,\,L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big)\end{aligned}$$

We first estimate the distance $\mathsf{L}\big(\,L_n(\mathbf{uk})\,,\,L_n(\mathbf{ul})\,\big)$. Indeed, since $\frac{1}{N_r(1-\gamma)} < \delta$, it follows that

$$\begin{aligned}\mathsf{L}\big(\,L_n(\mathbf{uk})\,,\,L_n(\mathbf{ul})\,\big) &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\leq 1}} \left|\int f\,d(L_n(\mathbf{uk})) - \int f\,d(L_n(\mathbf{ul}))\right|\\ &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\leq 1}} \left|\frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{uk})) - \frac{1}{n}\sum_{i=0}^{n-1} f(S^i(\mathbf{ul}))\right|\\ &\leq \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\leq 1}} \frac{1}{n}\sum_{i=0}^{n-1} |f(S^i(\mathbf{uk})) - f(S^i(\mathbf{ul}))|\\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} \mathsf{d}_\gamma\big(\,S^i(\mathbf{uk})\,,\,S^i(\mathbf{ul})\,\big)\\ &= \frac{1}{n}\sum_{i=0}^{n-1} \gamma^{|S^i(\mathbf{uk})\wedge S^i(\mathbf{ul})|}\\ &\leq \frac{1}{N_r}\sum_{i=0}^{n-1} \gamma^{n-i}\\ &\leq \frac{1}{N_r(1-\gamma)}\\ &< \delta\,,\end{aligned}$$

and we therefore conclude from (6.6) that

$$\mathsf{d}\big(\,UL_n(\mathbf{uk})\,,\,UL_n(\mathbf{ul})\,\big) < \frac{r}{2}\,. \tag{6.7}$$

Next, we estimate the distance $\mathsf{L}\big(\,L_n(\mathbf{ul})\,,\,L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big)$. We start by observing that if we fix $\mathbf{i}_0 \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$, then

$$\begin{aligned}\mathsf{L}(\mu,\nu) &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\leq 1}} \left|\int f\,d\mu - \int f\,d\nu\right|\\ &= \sup_{\substack{f:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(f)\leq 1}} \left|\int (f - f(\mathbf{i}_0))\,d\mu - \int (f - f(\mathbf{i}_0))\,d\nu\right|\\ &= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\leq 1\\ g(\mathbf{i}_0)=0}} \left|\int g\,d\mu - \int g\,d\nu\right| \end{aligned}\tag{6.8}$$

for all $\mu, \nu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. It follows from (6.8) that

$$
\begin{aligned}
\mathsf{L}\big(\, L_n(\mathbf{ul})\,,\, L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big)
&= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left| \int g\, d(L_n(\mathbf{ul})) - \int g\, d(L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}) \right| \\
&= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left| \frac{1}{n}\sum_{i=0}^{n-1} g(S^i(\mathbf{ul})) - \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=0}^{n+|\,\widehat{\mathbf{u}}\,|-1} g(S^i\overline{\mathbf{u}}) \right| \\
&\le \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left(\left| \frac{1}{n}\sum_{i=0}^{n-1} g(S^i(\mathbf{ul})) - \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=0}^{n-1} g(S^i\overline{\mathbf{u}}) \right| \right. \\
&\qquad\qquad\qquad\qquad \left. + \left| \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=n}^{n+|\,\widehat{\mathbf{u}}\,|-1} g(S^i\overline{\mathbf{u}}) \right| \right) \\
&\le \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} \left(\frac{|\,\widehat{\mathbf{u}}\,|}{n(n+|\,\widehat{\mathbf{u}}\,|)}\sum_{i=0}^{n-1} \|g\|_\infty + \frac{1}{n+|\,\widehat{\mathbf{u}}\,|}\sum_{i=n}^{n+|\,\widehat{\mathbf{u}}\,|-1} \|g\|_\infty \right) \\
&= \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} 2\frac{|\,\widehat{\mathbf{u}}\,|}{n+|\,\widehat{\mathbf{u}}\,|}\,\|g\|_\infty \\
&\le \sup_{\substack{g:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}\\ \mathrm{Lip}_\gamma(g)\le 1\\ g(\mathbf{i}_0)=0}} 2\frac{|\mathsf{V}|}{n}\,\|g\|_\infty\,. \qquad\qquad [\text{since } |\,\widehat{\mathbf{u}}\,| \le |\mathsf{V}|] \qquad (6.9)
\end{aligned}
$$

However, if $g : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ satisfies $\mathrm{Lip}_\gamma(g) \le 1$ and $g(\mathbf{i}_0) = 0$, then $|g(\mathbf{i})| = |g(\mathbf{i}) - g(\mathbf{i}_0)| \le \mathsf{d}_\gamma(\mathbf{i}, \mathbf{i}_0) \le 1$ for all $\mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$, whence $\|g\|_\infty \le 1$. It therefore follows from (6.9) that if $n \ge N_r$, $\mathbf{u} \in \Sigma_{\mathsf{G}}^{n}$ and $\mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$, then $\mathsf{L}\big(\, L_n(\mathbf{ul})\,,\, L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big) \le 2\frac{|\mathsf{V}|}{n} \le 2\frac{|\mathsf{V}|}{N_r} < \delta$, and we therefore conclude from (6.6) that

$$
\mathsf{d}\big(\, UL_n(\mathbf{ul})\,,\, L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big) < \frac{r}{2}\,. \tag{6.10}
$$

Finally, combining (6.7) and (6.10) shows that $\mathsf{L}\big(\, L_n(\mathbf{uk})\,,\, M_n(\mathbf{ul})\,\big) \le \mathsf{L}\big(\, L_n(\mathbf{uk})\,,\, L_n(\mathbf{ul})\,\big) + \mathsf{L}\big(\, L_n(\mathbf{ul})\,,\, L_{n+|\,\widehat{\mathbf{u}}\,|}\overline{\mathbf{u}}\,\big) < \frac{r}{2} + \frac{r}{2} = r$. □

We can now state and prove the first two key lemmas in this section.

Lemma 6.2. *Let (X, d) be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Fix $r > 0$. There is a positive integer N_r such that if $n \ge N_r$ and C is a subset of X, then we have*

$$
\begin{aligned}
\Big\{ \mathbf{u} \in \Sigma_{\mathsf{G}}^{n} \,\Big|\, UL_n[\mathbf{u}] \subseteq C \Big\} &\subseteq \Big\{ \mathbf{u} \in \Sigma_{\mathsf{G}}^{n} \,\Big|\, UM_n[\mathbf{u}] \subseteq B(C, r) \Big\}\,, \\
\Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UL_n\mathbf{i} \in C \Big\} &\subseteq \Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n\mathbf{i} \in B(C, r) \Big\}\,,
\end{aligned} \tag{6.11}
$$

and

$$
\begin{aligned}
\Big\{ \mathbf{u} \in \Sigma_{\mathsf{G}}^{n} \,\Big|\, UM_n[\mathbf{u}] \subseteq C \Big\} &\subseteq \Big\{ \mathbf{u} \in \Sigma_{\mathsf{G}}^{n} \,\Big|\, UL_u[\mathbf{u}] \subseteq B(C, r) \Big\}\,, \\
\Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n\mathbf{i} \in C \Big\} &\subseteq \Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UL_n\mathbf{i} \in B(C, r) \Big\}\,.
\end{aligned} \tag{6.12}
$$

Proof
It follows from Lemma 6.1 that there is a positive integer N_r such that if $n \geq N_r$, $\mathbf{u} \in \Sigma^n$ and $\mathbf{k}, \mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$, then $\mathsf{d}(\,UL_n(\mathbf{uk})\,,\,UM_n(\mathbf{ul})\,) < r$. In particular, this implies that if $n \geq N_r$ and $\mathbf{u} \in \Sigma^n$ with $UL_n(\mathbf{uk}) \subseteq C$, then

$$\begin{aligned} \operatorname{dist}\big(\,UM_n(\mathbf{ul})\,,\,C\,\big) &\leq \mathsf{d}\big(\,UM_n(\mathbf{ul})\,,\,UL_n(\mathbf{uk})\,\big) + \operatorname{dist}\big(\,UL_n(\mathbf{uk})\,,\,C\,\big) \\ &< r + 0 = r \end{aligned} \tag{6.13}$$

for all $\mathbf{k}, \mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$, and if $n \geq N_r$ and $\mathbf{u} \in \Sigma^n$ with $UM_n(\mathbf{ul}) \subseteq C$, then

$$\begin{aligned} \operatorname{dist}\big(\,UL_n(\mathbf{uk})\,,\,C\,\big) &\leq \mathsf{d}\big(\,UL_n(\mathbf{uk})\,,\,UM_n(\mathbf{ul})\,\big) + \operatorname{dist}\big(\,UM_n(\mathbf{ul})\,,\,C\,\big) \\ &< r + 0 = r \end{aligned} \tag{6.14}$$

for all $\mathbf{k}, \mathbf{l} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$ with $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{k})$ and $\mathrm{t}(\mathbf{u}) = \mathrm{i}(\mathbf{l})$. Inclusions (6.11) follow immediately from (6.13), and inclusions (6.12) follow immediately from (6.14). $\square$

Lemma 6.3. *Fix $r > 0$. There is a positive integer N_r such that if $n \geq N_r$ and C be a subset of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, then we have*

$$\begin{aligned} L_n^{-1}(C) &\subseteq M_n^{-1}(B(C,r))\,, \\ M_n^{-1}(C) &\subseteq L_n^{-1}(B(C,r))\,. \end{aligned}$$

Proof
This follows by applying the previous lemma to $X = \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and the map $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ defined by $U\mu = \mu$ for $\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. $\square$

The final result in this section is a simple continuity result about upper semi-continuous maps. This result is undoubtedly well-known. However, we have been unable to find a reference and for this reason we are including the short and simple proof.

Lemma 6.4. *Let X be a metric space and let $F : X \to \mathbb{R}$ be an upper semi-continuous function. Let $K_1, K_2, \ldots \subseteq X$ be non-empty compact subsets of X with $K_1 \supseteq K_2 \supseteq \ldots$. Then*

$$\inf_n \sup_{x \in K_n} F(x) = \sup_{x \in \bigcap_n K_n} F(x)\,.$$

Proof
First note that it is clear that $\inf_n \sup_{x \in K_n} F(x) \geq \sup_{x \in \cap_n K_n} F(x)$. We will now prove the reverse inequality, namely, $\inf_n \sup_{x \in K_n} F(x) \leq \sup_{x \in \cap_n K_n} F(x)$. Let $\varepsilon > 0$. For each n, we can choose $x_n \in K_n$ such that $F(x_n) \geq \sup_{x \in K_n} F(x) - \varepsilon$. Next, since K_n is compact for all n and $K_1 \supseteq K_2 \supseteq \ldots$, we can find a subsequence $(x_{n_k})_k$ and a point $x_0 \in \cap_n K_n$ such that $x_{n_k} \to x_0$. Also, since $K_{n_1} \supseteq K_{n_2} \supseteq \ldots$, we conclude that $\sup_{x \in K_{n_1}} F(x) \geq \sup_{x \in K_{n_2}} F(x) \geq \ldots$, whence $\inf_k \sup_{x \in K_{n_k}} F(x) = \limsup_k \sup_{x \in K_{n_k}} F(x)$. This implies that $\inf_n \sup_{x \in K_n} F(x) \leq \inf_k \sup_{x \in K_{n_k}} F(x) = \limsup_k \sup_{x \in K_{n_k}} F(x) \leq \limsup_k F(x_{n_k}) + \varepsilon$. However, since $x_{n_k} \to x_0$, we deduce from the upper semi-continuity of the function F, that $\limsup_k F(x_{n_k}) \leq F(x_0)$. Consequently $\inf_n \sup_{x \in K_n} F(x) \leq \limsup_k F(x_{n_k}) + \varepsilon \leq F(x_0) + \varepsilon \leq \sup_{x \in \cap_n K_n} F(x) + \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives the desired result. $\square$

7. Proofs. The measures Π and Π_n

In this section we introduce two technical auxiliary measures, namely, the measures Π and Π_n; see definitions (7.2) and (7.3) below. The main result is this section is Theorem 7.2 showing that the sequence $(\Pi_n)_n$ has the large deviation property. Theorem 7.2 plays a major role in the in the proof of Theorem 8.3 in Section 8. We start by introducing some notation. We first introduce the two main auxiliary measures Π and Π_n.

The measure Π. Let $B = (b_{\mathsf{i},\mathsf{j}})_{\mathsf{i},\mathsf{j}\in\mathsf{V}}$ denote the matrix defined by

$$b_{\mathsf{i},\mathsf{j}} = |\mathsf{E}_{\mathsf{i},\mathsf{j}}|\,.$$

We denote the spectral radius of B by λ. Since $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ is strongly connected, we conclude that the matrix B is irreducible, and it therefore follows from the Perron-Frobenius theorem that there is a unique right eigen-vector $\mathbf{u} = (u_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of B with eigen-value λ and a unique left eigen-vector $\mathbf{v} = (v_{\mathsf{i}})_{\mathsf{i}\in\mathsf{V}}$ of B with eigen-value λ, i.e.

$$\begin{aligned} \mathbf{u}B &= \lambda\mathbf{u}\,,\\ B\mathbf{v} &= \lambda\mathbf{v}\,, \end{aligned} \tag{7.1}$$

with $u_{\mathsf{i}}, v_{\mathsf{i}} > 0$ for all i such that $\sum_{\mathsf{i}} u_{\mathsf{i}}v_{\mathsf{i}} = 1$ and $\sum_{\mathsf{i}} u_{\mathsf{i}} = 1$. For $\mathsf{e} \in \mathsf{V}$, write $\pi_{\mathsf{e}} = v_{\mathsf{i}(\mathsf{e})}^{-1}\, v_{\mathsf{t}(\mathsf{e})}\,\lambda^{-1}$. A simple calculation shows that $\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{i}}} \pi_{\mathsf{e}} = 1$ for all i and that $\sum_{\mathsf{i}}\sum_{\mathsf{e}\in\mathsf{E}_{\mathsf{i},\mathsf{j}}} u_{\mathsf{i}}\, v_{\mathsf{i}}\, \pi_{\mathsf{e}} = u_{\mathsf{j}}\, v_{\mathsf{j}}$ for all j. It follows from this that there is a unique Borel probability measure $\Pi \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ such that

$$\begin{aligned} \Pi[\mathbf{i}] &= u_{\mathsf{i}(\mathsf{e}_1)}\, v_{\mathsf{i}(\mathsf{e}_1)}\, \pi_{\mathsf{e}_1}\, \cdots\, \pi_{\mathsf{e}_n}\\ &= u_{\mathsf{i}(\mathsf{e}_1)}\, v_{\mathsf{t}(\mathsf{e}_n)}\, \lambda^{-n}\\ &= u_{\mathsf{i}(\mathbf{i})}\, v_{\mathsf{t}(\mathbf{i})}\, \lambda^{-n} \end{aligned} \tag{7.2}$$

for all $\mathbf{i} = \mathsf{e}_1 \ldots \mathsf{e}_n \in \Sigma_{\mathsf{G}}^{*}$.

The measure Π_n. Finally, for a positive integer n, we define the probability measures $\Pi_n \in \mathcal{P}(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}))$ by

$$\Pi_n = \Pi \circ M_n^{-1}\,; \tag{7.3}$$

recall that the map $M_n : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is defined in (6.1).

We now turn towards the proof of the main result in this section, namely, Theorem 7.2 showing that the sequence $(\Pi_n)_n$ has the large deviation property. The proof of Theorem 7.2 is based on a large deviation theorem by Orey & Pelikan [OrPe1,OrPe2] (see also [JiQiQi, Remark 7.2.2]). In particular, Orey & Pelikan [OrPe1,OrPe2] prove that various sequences of Gibbs measures satisfy a large deviation principle, see Theorem 7.1 below. However, we begin by stating the definition of the large deviation principle.

Definition. *Let X be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on X. Let $(a_n)_n$ be a sequence of positive numbers with $a_n \to \infty$ and let $I : X \to [0, \infty]$ be a lower semi-continuous function with compact level sets. The sequence $(P_n)_n$ is said to have the large deviation property with constants $(a_n)_n$ and rate function I if the following two condistions hold:*

(i) *For each closed subset K of X, we have*

$$\limsup_n \frac{1}{a_n} \log P_n(K) \le -\inf_{x\in K} I(x)\,;$$

(ii) *For each open subset G of X, we have*

$$\liminf_n \frac{1}{a_n} \log P_n(G) \ge -\inf_{x\in G} I(x)\,.$$

Theorem 7.1. [OrPe1,OrPe2].

(1) *Let $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ be a Hölder continous function. Let μ_Φ be the Gibbs state of Φ and write $\mu_{\Phi,n} = \mu_\Phi \circ L_n^{-1}$. Define $I_\Phi : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to [0, \infty]$ by*

$$I_\Phi(\mu) = \begin{cases} P(\Phi) - \int \Phi\, d\mu - h(\mu) & \text{for } \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}});\\ \infty & \text{for } \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}). \end{cases}$$

Then the sequence $(\mu_{\Phi,n})_n$ has the large deviation deviation property with respect to the sequence $(n)_n$ and rate function I_Φ.

(2) *Write $\Gamma_n = \Pi \circ L_n^{-1}$. Define $I : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to [0,\infty]$ by*

$$I(\mu) = \begin{cases} \log\lambda - h(\mu) & \text{for } \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}); \\ \infty & \text{for } \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}). \end{cases}$$

Then the sequence $(\Gamma_n)_n$ has the large deviation deviation property with respect to the sequence $(n)_n$ and rate function I.

Proof
(1) This follows from [OrPe1,OrPe2].
(2) We use the notation from Part (1), namely, if $\Phi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ is a Hölder continous function, then we let μ_Φ denote the Gibbs state of Φ and we write $\mu_{\Phi,n}$ and I_Φ for the measure and function defined in the statement of part (1) of the theorem. Let $\mathcal{O} : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$ denote the zero-function, i.e. $\mathcal{O}(\mathbf{i}) = 0$ for all $\mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}}$. Noticing that $\mu_{\mathcal{O}} = \Pi$ (by [Wa, Theorem 8.10]) and $P(\mathcal{O}) = \log\lambda$ (indeed, the variational principle implies that $P(\mathcal{O}) = \sup_{\mu\in\Sigma_{\mathsf{G}}^{\mathbb{N}}} h(\mu)$, and we deduce from [Wa, Theorem 8.10] that $\sup_{\mu\in\Sigma_{\mathsf{G}}^{\mathbb{N}}} h(\mu) = h(\Pi)$ and from [Wa, Theorem 7.13] that $h(\Pi) = \log\lambda$; combing these results show that $P(\mathcal{O}) = \sup_{\mu\in\Sigma_{\mathsf{G}}^{\mathbb{N}}} h(\mu) = \mu(\Pi) = \log\lambda$), we conclude that $\mu_{\mathcal{O},n} = \mu_{\mathcal{O}} \circ L_n^{-1} = \Pi \circ L_n^{-1} = \Gamma_n$ and $I_{\mathcal{O}} = I$, and the result therefore follows immediately from (1). $\square$

We can now state and prove the main result in this section.

Theorem 7.2. *Define $I : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to [0,\infty]$ by*

$$I(\mu) = \begin{cases} \log\lambda - h(\mu) & \text{for } \mu \in \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}); \\ \infty & \text{for } \mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}). \end{cases}$$

Then the sequence $(\Pi_n)_n$ has the large deviation deviation property with respect to the sequence $(n)_n$ and rate function I.

Proof
We will use the same notation as in Theorem 7.1, i.e. for a positive integer n, we write $\Gamma_n = \Pi \circ L_n^{-1}$. We now prove the following two claims.

Claim 1. For each closed subset K of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, we have

$$\limsup_n \frac{1}{n} \log \Pi_n(K) \le -\inf_{\mu\in K} I(\mu)\,.$$

Proof of Claim 1. Let K be a close subset of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Now, observe that it follows from Lemma 6.3 that for each $r > 0$, we can find a positive integer N_r such that for all $n \ge N_r$ we have

$$\Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, M_n\mathbf{i} \in K \Big\} \subseteq \Big\{ \mathbf{i} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, L_n\mathbf{i} \in B(K,r) \Big\}\,.$$

This clearly implies that for all $n \ge N_r$ we have

$$\begin{aligned} \Pi_n(K) &\le \Gamma_n\big(\, B(K,r)\,\big) \\ &\le \Gamma_n\big(\, \overline{B(K,r)}\,\big)\,. \end{aligned} \tag{7.4}$$

Next, since $\overline{B(K,r)}$ is a closed set and since it follows from Theorem 7.1 that the sequence $(\Gamma_n)_n$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I, we conclude from (7.4) that

$$\begin{aligned} \limsup_n \frac{1}{n} \log \Pi_n(K) &\le \limsup_n \frac{1}{n} \log \Gamma_n(\, \overline{B(K,r)}\,) \\ &\le -\inf_{\mu\in\overline{B(K,r)}} I(\mu) \end{aligned}$$

for all $r > 0$. Hence

$$\begin{aligned}\limsup_n \frac{1}{n}\log \Pi_n(K) &\le \inf_{r>0}\left(-\inf_{\mu\in\overline{B(K,r)}} I(\mu)\right)\\ &= \inf_{r>0}\sup_{\mu\in\overline{B(K,r)}} -I(\mu)\\ &= \inf_n \sup_{\mu\in\overline{B(K,\frac{1}{n})}} -I(\mu)\,. \end{aligned}\tag{7.5}$$

However, it follows from Theorem 7.1 that I is a rate function and therefore, in particular, lower semi-continuous. We deduce from this that the function $-I$ is upper semi-continuous. Since the sets $K_n = \overline{B(K,\frac{1}{n})}$ are compact (because they are closed subsets of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ and $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is compact) with $K_1 \supseteq K_2 \supseteq K_3 \ldots$, we therefore conclude from Lemma 6.4 and the fact that $-I$ is upper semi-continuous that

$$\begin{aligned}\inf_n \sup_{\mu\in\overline{B(K,\frac{1}{n})}} -I(\mu) &= \inf_n \sup_{\mu\in K_n} -I(\mu)\\ &= \sup_{\mu\in\bigcap_n K_n} -I(\mu)\end{aligned}\tag{7.6}$$

Next, observing that $\cap_n K_n = \cap_n \overline{B(K,\frac{1}{n})} = K$ (because K is closed), we deduce from (7.6) that

$$\begin{aligned}\inf_n \sup_{\mu\in\overline{B(K,\frac{1}{n})}} -I(\mu) &= \sup_{\mu\in K} -I(\mu)\\ &= -\inf_{\mu\in K} I(\mu)\,.\end{aligned}\tag{7.7}$$

Finally. combining (7.5) and (7.7) shows that $\limsup_n \frac{1}{n}\log \Pi_n(K) \le -\inf_{\mu\in K} I(\mu)$. This completes the proof of Claim 1.

Claim 2. For each open subset G of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, we have

$$\liminf_n \frac{1}{n}\log \Pi_n(G) \ge -\inf_{\mu\in G} I(\mu)\,.$$

Proof of Claim 2. Let G be an open subset of $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. For each $r > 0$, we will write $I(G,r) = \{\mu \in \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \,|\, \operatorname{dist}(\mu, \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \setminus G) > r\}$. Now, observe that it follows from Lemma 6.3 that for each $r > 0$, we can find a positive integer N_r such that for all $n \ge N_r$ we have

$$\left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, L_n\mathbf{i}\in I(G,r)\right\} \subseteq \left\{\mathbf{i}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, M_n\mathbf{i}\in B(\,I(G,r)\,,\,\tfrac{r}{2}\,)\right\}.$$

This clearly implies that for all $n \ge N_r$ we have

$$\Gamma_n(\,I(G,r)\,) \le \Pi_n\big(\,B(\,I(G,r)\,,\,\tfrac{r}{2}\,)\,\big)\tag{7.8}$$

However, since it is easily seen that $B(\,I(G,r)\,,\,\frac{r}{2}) \subseteq G$, it follows that $\Pi_n\big(\,B(\,I(G,r)\,,\,\frac{r}{2})\,\big) \le \Pi_n(G)$, and (7.8) therefore shows that for all $n \ge N_r$ we have

$$\Gamma_n(\,I(G,r)\,) \le \Pi_n(G)\,.\tag{7.9}$$

Next, since $I(G,r)$ is an open set and since it follows from Theorem 7.1 that the sequence $(\Gamma_n)_n$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I, we conclude from (7.9) that

$$\begin{aligned}\liminf_n \frac{1}{n}\log \Pi_n(G) &\ge \liminf_n \frac{1}{n}\log \Gamma_n(\,I(G,r)\,)\\ &\ge -\inf_{\mu\in I(G,r)} I(\mu)\end{aligned}$$

for all $r > 0$. Hence

$$\begin{aligned}\liminf_n \frac{1}{n} \log \Pi_n(G) &\geq \sup_{r>0} \left(-\inf_{\mu \in I(G,r)} I(\mu) \right) \\ &= -\inf_{r>0} \inf_{\mu \in I(G,r)} I(\mu) \\ &= -\inf_{\mu \in \bigcup_{r>0} I(G,r)} I(\mu) \,. \end{aligned} \tag{7.10}$$

Finally, since G is open, it follows easily that $\cup_{r>0} I(G,r) = G$, and (7.10) therefore implies that $\liminf_n \frac{1}{n} \log \Pi_n(G) \geq -\inf_{\mu \in \cup_{r>0} I(G,r)} I(\mu) = -\inf_{\mu \in G} I(\mu)$. This completes the proof of Claim 2.

the desired result follows immediately from Claim 1 and Claim 2. □

8. Proofs. The modified multifractal pressure

In this section we introduce our main technical tool, namely, the modified multifractal pressure; see definition (8.1) below. The two main results in this section are Theorem 8.3 and Theorem 8.4. Theorem 8.3 applies Theorem 7.2 (saying that the sequence $(\Pi_n)_n$ satisfies the large deviation principle) to establish a variational principle for the modified multifractal pressure and Theorem 8.4 shows that the multifractal pressure and the modified multifractal pressure are 'asymptotically" the same.

We start be defined the modified multifractal pressure. For $C \subseteq X$, we define the modified lower and upper mutifractal pressure of φ associated with the space X and the map U and by

$$\begin{aligned} \underline{Q}_C^U(\varphi) &= \liminf_n \; \frac{1}{n} \; \log \sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{i}] \subseteq C}} \; \sup_{\mathbf{u} \in [\mathbf{i}]} \; \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \,, \\ \overline{Q}_C^U(\varphi) &= \limsup_n \; \frac{1}{n} \; \log \sum_{\substack{\mathbf{i} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{i}] \subseteq C}} \; \sup_{\mathbf{u} \in [\mathbf{i}]} \; \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \,. \end{aligned} \tag{8.1}$$

In order to establish a variational principle for the modified multifractal pressure, we introduce the following notation. For a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$, we define $F_\varphi : \mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to \mathbb{R}$ by

$$F_\varphi(\mu) = \int \varphi \, d\mu \,.$$

Observe that since φ is bounded, i.e. $\|\varphi\|_\infty < \infty$, we conclude that $\|F_\varphi\|_\infty \leq \|\varphi\|_\infty < \infty$. In addition, for a positive integer n, define the probability measure $Q_{\varphi,n} \in \mathcal{P}\big(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ by

$$Q_{\varphi,n}(E) = \frac{\int_E \exp(nF_\varphi) \, d\Pi_n}{\int \exp(nF_\varphi) \, d\Pi_n} \quad \text{for Borel subsets } E \text{ of } \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}});$$

recall, that the measure Π_n is defined in (7.3).

We now turn towards the proof of the first main result in this section, namely, Theorem 8.2 providing a variational principle for the modified multifractal pressure. The proof of Theorem 8.2 is based on large deviation theory. In particular, we will use the fact that the sequence $(\Pi_n)_n$ satisfies the large deviation principle together with Varadhan's [Va] large deviation theorem (Theorem 8.1.(1) below), and a non-trivial application of this (namely Theorem 8.1.(2) below) providing first order asymptotics of certain "Boltzmann distributions". Recall, that the notion of a large deviation principle is defined in Section 7.

Theorem 8.1. *Let X be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on X. Assume that the sequence $(P_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function I. Let $F : X \to \mathbb{R}$ be a continuous function satisfying the following two conditions:*

(i) *For all n, we have*

$$\int \exp(a_n F)\, dP_n < \infty\,.$$

(ii) *We have*

$$\lim_{M\to\infty} \limsup_{n} \frac{1}{a_n} \log \int_{\{M\leq F\}} \exp(a_n F)\, dP_n = -\infty\,.$$

(Observe that the Conditions (i)–(ii) are satisfied if F is bounded.) Then the following statements hold.

(1) *We have*

$$\lim_{n} \frac{1}{a_n} \log \int \exp(a_n F)\, dP_n = -\inf_{x\in X}(I(x) - F(x))\,.$$

(2) *For each n define a probability measure Q_n on X by*

$$Q_n(E) = \frac{\int_E \exp(a_n F)\, dP_n}{\int \exp(a_n F)\, dP_n}\,.$$

Then the sequence $(Q_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function $(I - F) - \inf_{x\in X, I(x)<\infty}(I(x) - F(x))$.

Proof
Statement (1) follows from [El, Theorem II.7.1] or [DeZe, Theorem 4.3.1], and statement (2) follows from [El, Theorem II.7.2]. □

Before stating and proving Theorem 8.3, we establish the following auxiliary result.

Theorem 8.2. *Let X be a metric space and let $U : \mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of X. Fix a continuous function $\varphi : \Sigma_{\mathsf{G}}^{\mathbb{N}} \to \mathbb{R}$. Then there is a constant c such that for all positive integers n, we have*

$$\sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \leq c\, \lambda^n\, Q_{\varphi,n}\Big(\{U \in C\}\Big) \int \exp(nF_\varphi)\, d\Pi_n\,,$$

$$\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} \sup_{\mathbf{u}\in[\mathbf{k}]} \exp \sum_{i=0}^{n-1} \varphi S^i \mathbf{u} \geq \frac{1}{c}\, \lambda^n\, Q_{\varphi,n}\Big(\{U \in C\}\Big) \int \exp(nF_\varphi)\, d\Pi_n\,.$$

Proof
For each positive integer n and each $\mathbf{i} \in \Sigma_{\mathsf{G}}^{n}$, we write $s_{\mathbf{i}} = \sup_{\mathbf{u}\in[\mathbf{i}]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u}$ for sake of brevity. Let C be a subset of X. For each positive integer n, we clearly have

$$\int\limits_{\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq C\right\}} s_{\mathbf{i}|n}\, d\Pi(\mathbf{i})$$

$$= \sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}} \int\limits_{[\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\, UM_n[\mathbf{j}|n]\subseteq C\right\}} s_{\mathbf{i}|n}\, d\Pi(\mathbf{i})$$

$$
\begin{aligned}
&= \sum_{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\\
&= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ [\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}\neq\varnothing}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\,.
\end{aligned}
\tag{8.3}
$$

Now observe that if $\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}$ with $[\mathbf{k}]\cap\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq C\}\neq\varnothing$, then a simple argument shows that $UM_n[\mathbf{k}]\subseteq C$. We conclude from this that

$$
\begin{aligned}
&\sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ [\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}\neq\varnothing}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ [\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}\neq\varnothing\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\,.
\end{aligned}
\tag{8.4}
$$

Combining (8.3) and (8.4) gives

$$
\begin{aligned}
&\int_{\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}} s_{\mathbf{i}|n}\, d\Pi(\mathbf{i})\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ [\mathbf{k}]\,\cap\,\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}\neq\varnothing\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\,.\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\,.
\end{aligned}
\tag{8.5}
$$

However, if $\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}$ with $UM_n[\mathbf{k}]\subseteq C$, then it is clear that $[\mathbf{k}]\subseteq\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq C\}$, whence $[\mathbf{k}]\cap\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,|\,UM_n[\mathbf{j}|n]\subseteq C\}=[\mathbf{k}]$. This and (8.5) now imply that

$$
\begin{aligned}
&\int_{\left\{\mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}}\,\middle|\,UM_n[\mathbf{j}|n]\subseteq C\right\}} s_{\mathbf{i}|n}\, d\Pi(\mathbf{i})\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, \Pi\Big([\mathbf{k}] \;\cap\; \Big\{ \mathbf{j}\in\Sigma_{\mathsf{G}}^{\mathbb{N}} \,\Big|\, UM_n[\mathbf{j}|n]\subseteq C \Big\} \Big)\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, \Pi\big([\mathbf{k}] \big)\\
&\qquad= \sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}}\, u_{\mathrm{i}(\mathbf{k})}\, v_{\mathrm{t}(\mathbf{k})}\, \lambda^{-n}\,.
\end{aligned}
\tag{8.6}
$$

It follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa]) that there is a constant $c_0 > 0$ such that if $n \in \mathbb{N}$, $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$ and $\mathbf{u}, \mathbf{v} \in [\mathbf{i}]$, then $\frac{1}{c} \le \frac{\exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u}}{\exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{v}} \le c$. In particular, this implies that for all $n \in \mathbb{N}$ and for all $\mathbf{i} \in \Sigma_{\mathsf{G}}^n$, we have

$$\frac{1}{c_0} \exp \sum_{k=0}^{n-1} \varphi S^k \bar{\mathbf{i}} \le s_{\mathbf{i}} \le c_0 \exp \sum_{k=0}^{n-1} \varphi S^k \bar{\mathbf{i}} \,. \tag{8.7}$$

We can also find a constant $c_1 > 0$ such that $\frac{1}{c_1} \le u_{\mathsf{i}} v_{\mathsf{i}} \le c_1$ for all i. Now put $c = c_0 c_1$.

Claim 1. For all positive integers n, we have

$$\sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} s_{\mathbf{k}} \le c\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(nF_\varphi(M_n \mathbf{i})\right)\, d\Pi(\mathbf{i})\,, \tag{8.8}$$

$$\sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} s_{\mathbf{k}} \ge \frac{1}{c}\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(nF_\varphi(M_n \mathbf{i})\right)\, d\Pi(\mathbf{i})\,. \tag{8.9}$$

Proof of Claim 1. It follows from (8.6) and (8.7) that if n is a positive integer, then we have

$$\begin{aligned}
\sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} s_{\mathbf{k}} &\le c\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(\sum_{k=0}^{n-1} \varphi S^k \left(\overline{\mathbf{i}|n}\right)\right) d\Pi(\mathbf{i}) \\
&= c\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(n \int \varphi \, d(M_n \mathbf{i})\right) d\Pi(\mathbf{i}) \\
&= c\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(nF_\varphi(M_n \mathbf{i})\right)\, d\Pi(\mathbf{i})\,.
\end{aligned}$$

This proves inequality (8.8). Inequality (8.9) is proved similarly. This completes the proof of Claim 1.

Claim 2. For all positive integers n, we have $\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,|\, UM_n[\mathbf{j}|n] \subseteq C\} = \{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,|\, UM_n \mathbf{j} \subseteq C\}$.
Proof of Claim 2. This is easily seen; however, for the convenience of the reader we have decided to state the result explicitly. This completes the proof of Claim 2.

For all positive integers n, we now deduce from Claim 1 and Claim 2 that

$$\begin{aligned}
\sum_{\substack{\mathbf{k} \in \Sigma_{\mathsf{G}}^n \\ UM_n[\mathbf{k}] \subseteq C}} s_{\mathbf{k}} &\le c\,\lambda^n \int\limits_{\left\{\mathbf{j} \in \Sigma_{\mathsf{G}}^{\mathbb{N}} \,\middle|\, UM_n[\mathbf{j}|n] \subseteq C\right\}} \exp\left(nF_\varphi(M_n \mathbf{i})\right)\, d\Pi(\mathbf{i}) \\
&= c\,\lambda^n \int\limits_{\left\{UM_n \in C\right\}} \exp\left(nF_\varphi(M_n \mathbf{i})\right)\, d\Pi(\mathbf{i}) \\
&= c\,\lambda^n \int\limits_{\left\{U \in C\right\}} \exp\left(nF_\varphi\right)\, d\Pi_n \\
&= c\,\lambda^n \; Q_{\varphi,n}\Big(\{U \in C\}\Big) \int \exp\left(nF_\varphi\right)\, d\Pi_n\,.
\end{aligned}$$

Similarly, we prove that for all positive integers n, we have

$$\sum_{\substack{\mathbf{k}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{k}]\subseteq C}} s_{\mathbf{k}} \geq \frac{1}{c}\,\lambda^{n}\; Q_{\varphi,n}\Big(\{U\in C\}\Big)\;\int \exp\left(nF_{\varphi}\right)\, d\Pi_{n}\,.$$

This completes the proof of Theorem 8.2. $\square$

Theorem 8.3. The variational principle for the modified multifractal pressure. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$.*

(1) *If G is an open subset of X with $U^{-1}G\cap\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\neq\varnothing$, then*

$$\underline{Q}_{G}^{U}(\varphi)\geq \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}\left(h(\mu)+\int\varphi\,d\mu\right). \tag{8.10}$$

(2) *If K is a closed subset of X with $U^{-1}K\cap\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\neq\varnothing$, then*

$$\overline{Q}_{K}^{U}(\varphi)\leq \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in K}}\left(h(\mu)+\int\varphi\,d\mu\right). \tag{8.11}$$

Proof
We introduce the simplified notation from the proof of Theorem 8.2, i.e. for each positive integer n and each $\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}$, we write $s_{\mathbf{i}}=\sup_{\mathbf{u}\in[\mathbf{i}]}\exp\sum_{k=0}^{n-1}\varphi S^{k}\mathbf{u}$. First note that it follows immediately from Theorem 8.2 that

$$\begin{aligned}
\liminf_{n}\frac{1}{n}\log\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{i}]\subseteq G}} s_{\mathbf{i}} &\geq \log\lambda\,+\,\liminf_{n}\frac{1}{n}\log Q_{\varphi,n}\Big(\{U\in G\}\Big)\\
&\qquad\qquad +\,\liminf_{n}\frac{1}{n}\log\int\exp\left(nF_{\varphi}\right)\,d\Pi_{n}\,,\\
\limsup_{n}\frac{1}{n}\log\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^{n}\\ UM_n[\mathbf{i}]\subseteq K}} s_{\mathbf{i}} &\leq \log\lambda\,+\,\limsup_{n}\frac{1}{n}\log Q_{\varphi,n}\Big(\{U\in K\}\Big)\\
&\qquad\qquad +\,\limsup_{n}\frac{1}{n}\log\int\exp\left(nF_{\varphi}\right)\,d\Pi_{n}\,.
\end{aligned} \tag{8.12}$$

Define $I:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to[0,\infty]$ by

$$I(\mu)=\begin{cases}\log\lambda-h(\mu) & \text{for } \mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}});\\ \infty & \text{for } \mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\setminus\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}}).\end{cases}$$

Next, we observe that it follows from Theorem 7.2 that the sequence $(\Pi_n)_n\subseteq\mathcal{P}\big(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I. We therefore conclude from Part (1) of Theorem 8.1 that

$$\begin{aligned}
\lim_{n}\frac{1}{n}\log\int\exp\left(nF_{\varphi}\right)\,d\Pi_{n} &= -\inf_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(I(\nu)-F_{\varphi}(\nu)\right)\\
&= -\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}\left(I(\nu)-F_{\varphi}(\nu)\right).
\end{aligned} \tag{8.13}$$

Note that it follows that

$$(I-F_\varphi)-\inf_{\substack{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ I(\nu)<\infty}}(I(\nu)-F_\varphi(\nu))=(I-F_\varphi)-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\,; \tag{8.14}$$

indeed, if $\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, then it is clear from the definition of I that $I(\nu)=\infty$ if and only if $\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\setminus\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, i.e. $\{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,|\,I(\nu)<\infty\}=\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, whence $\inf_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}),I(\nu)<\infty}(I(\nu)-F_\varphi(\nu))=\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))$, and equality (8.14) follows immediately from this. Also, since the sequence $(\Pi_n)_n\subseteq\mathcal{P}\big(\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)$ has the large deviation property with respect to the sequence $(n)_n$ and rate function I, we conclude from Part (2) of Theorem 8.1 that the sequence $(Q_{\varphi,n})_n$ has the large deviation property with respect to the sequence $(n)_n$ and rate function $(I-F_\varphi)-\inf_{\nu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}}),I(\nu)<\infty}(I(\nu)-F_\varphi(\nu))=(I-F_\varphi)-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))$ (where we have used (8.14)). As the set $\{U\in G\}=U^{-1}G$ is open and the set $\{U\in K\}=U^{-1}K$ is closed, it therefore follows from the large deviation property that

$$\begin{aligned}
&\limsup_n\frac{1}{n}\log Q_{\varphi,n}\Big(\{U\in G\}\Big)\\
&\qquad\geq-\inf_{\substack{\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}\left((I(\mu)-F_\varphi(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\right).\\
&\qquad=-\inf_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}\left((I(\mu)-F_\varphi(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\right).\\
&\limsup_n\frac{1}{n}\log Q_{\varphi,n}\Big(\{U\in K\}\Big)\\
&\qquad\leq-\inf_{\substack{\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in K}}\left((I(\mu)-F_\varphi(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\right)\\
&\qquad=-\inf_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in K}}\left((I(\mu)-F_\varphi(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\right).
\end{aligned} \tag{8.15}$$

Combining (8.12). (8.13) and (8.15) now yields

$$\begin{aligned}
\limsup_n\frac{1}{n}\log\sum_{\substack{\mathbf{i}\in\Sigma_{\mathsf{G}}^n\\ UM_n[\mathbf{i}]\subseteq G}}s_{\mathbf{i}}\geq\log\lambda\,+\,&\limsup_n\frac{1}{n}\log Q_{\varphi,n}\Big(\{U\in G\}\Big)\\
&\qquad+\,\limsup_n\frac{1}{n}\log\int\exp\left(nF_\varphi\right)\,d\Pi_n\\
&\geq\log\lambda\\
&\qquad-\inf_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}\left((I(\mu)-F_\varphi(\mu))-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\right)\\
&\qquad\qquad-\inf_{\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})}(I(\nu)-F_\varphi(\nu))\\
&=\log\lambda\,+\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}(F_\varphi(\mu)-I(\mu))\\
&=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G}}\left(h(\mu)+\int\varphi\,d\mu\right).
\end{aligned}$$

This completes the proof of inequality (8.10). Inequality (8.11) is proved similarly. □

We now turn towards the second main result in this section, namely, Theorem 8.4 showing that the multifractal pressure and the modified multifractal pressure are 'asymptotically" the same.

Theorem 8.4. *Let X be a metric space and let $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous with respect to the weak topology. Let $C\subseteq X$ be a subset of X and $r>0$. Fix a continuous function $\varphi:\Sigma_{\mathsf{G}}^{\mathbb{N}}\to\mathbb{R}$. Then we have*

$$\begin{aligned}\underline{P}_C^U(\varphi)\le\underline{Q}_{B(C,r)}^U(\varphi)\,,\quad \underline{Q}_C^U(\varphi)\le\underline{P}_{B(C,r)}^U(\varphi)\,,\\ \overline{P}_C^U(\varphi)\le\overline{Q}_{B(C,r)}^U(\varphi)\,,\quad \overline{Q}_C^U(\varphi)\le\overline{P}_{B(C,r)}^U(\varphi)\,.\end{aligned}$$

Proof
This follows immediately from Lemma 6.2. □

9. Proof of Theorem 4.4

The purpose of this section is to prove Theorem 4.4.

Proof of Theorem 4.4
(1) We must prove the following two inequalities, namely,

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)\le\inf_{r>0}\,\underline{P}_{B(C,r)}^U(\varphi)\,,\tag{9.1}$$

$$\inf_{r>0}\,\overline{P}_{B(C,r)}^U(\varphi)\le\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right).\tag{9.2}$$

Proof of (9.1). Since $B(C,r)$ is open with $\overline{C}\subseteq B(C,r)$, we conclude from Theorem 8.3 that

$$\begin{aligned}\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)&\le\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in B(C,r)}}\left(h(\mu)+\int\varphi\,d\mu\right)\\ &\le\underline{Q}_{B(C,r)}^U(\varphi)\,.\end{aligned}\tag{9.3}$$

Taking infimum over all $r>0$ in (9.3) gives

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)\le\inf_{r>0}\,\underline{Q}_{B(C,r)}^U(\varphi)\,.\tag{9.4}$$

Next, we note that it follows from Theorem 8.4 that $\underline{Q}_{B(C,r)}^U(\varphi)\le\underline{P}_{B(\,B(C,r)\,,\,r\,)}^U(\varphi)$. Combining this inequality with and (9.4) and using the fact that $B(\,B(C,r)\,,\,r\,)\subseteq B(C,2r)$, we now conclude from Theorem 8.4 that

$$\begin{aligned}\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)&\le\inf_{r>0}\,\underline{Q}_{B(C,r)}^U(\varphi)\\ &\le\inf_{r>0}\,\underline{P}_{B(\,B(C,r)\,,\,r\,)}^U(\varphi)\qquad\text{[by Theorem 8.4]}\\ &\le\inf_{r>0}\,\underline{P}_{B(C,2r)}^U(\varphi)\\ &\le\inf_{s>0}\,\underline{P}_{B(C,s)}^U(\varphi)\,.\end{aligned}$$

This completes the proof of inequality (9.1).

Proof of (9.2). Since $B(\,B(C,r)\,,\,r\,)\subseteq B(C,2r)\subseteq\overline{B(C,2r)}$ and $\overline{B(C,2r)}$ is closed, we conclude from Theorem 8.3 and Theorem 8.4 that

$$\begin{aligned}
\inf_{r>0}\ \overline{P}^{U}_{B(C,r)}(\varphi) &\leq \inf_{r>0}\ \overline{Q}^{U}_{B(\,B(C,r)\,,\,r\,)}(\varphi) && \text{[by Theorem 8.4]}\\
&\leq \inf_{r>0}\ \overline{Q}^{U}_{\overline{B(C,2r)}}(\varphi)\\
&\leq \inf_{r>0}\ \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{B(C,2r)}}}\left(h(\mu)+\int\varphi\,d\mu\right). && \text{[by Theorem 8.3]}
\end{aligned}$$

Letting $U_S:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ denote the restriction of U to $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, the above inequality can be written as

$$\begin{aligned}
\inf_{r>0}\ \overline{P}^{U}_{B(C,r)}(\varphi) &\leq \inf_{r>0}\ \sup_{\mu\in U_S^{-1}\overline{B(C,2r)}}\left(h(\mu)+\int\varphi\,d\mu\right)\\
&= \inf_{n}\ \sup_{\mu\in U_S^{-1}\overline{B(C,\frac{1}{n})}}\left(h(\mu)+\int\varphi\,d\mu\right). \qquad (9.5)
\end{aligned}$$

Next, note that since $\overline{B(C,\frac{1}{n})}$ is closed and U_S is continuous, the set $U_S^{-1}\overline{B(C,\frac{1}{n})}$ is a closed subset of $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. As $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is compact, we therefore deduce that $U_S^{-1}\overline{B(C,\frac{1}{n})}$ is compact. Also, note that it follows from [Wa, Theorem 8.2] that the entropy function $h:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ is upper semi-continuous. We conclude from this that the map $F:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ defined by $F(\mu)=h(\mu)+\int\varphi\,d\mu$ is upper semi-continuous. Finally, since the sets $K_n=U_S^{-1}\overline{B(C,\frac{1}{n})}$ are compact with $K_1\supseteq K_2\supseteq K_3\supseteq\dots$ and F is upper semi-continuous, we deduce from Lemma 6.4 that

$$\begin{aligned}
\inf_{n}\ \sup_{\mu\in U_S^{-1}\overline{B(C,\frac{1}{n})}}\left(h(\mu)+\int\varphi\,d\mu\right) &= \inf_{n}\ \sup_{\mu\in K_n}\ F(\mu)\\
&= \sup_{\mu\in\bigcap_n K_n} F(\mu)\\
&= \sup_{\mu\in\bigcap_n U_S^{-1}\overline{B(C,\frac{1}{n})}}\left(h(\mu)+\int\varphi\,d\mu\right). \qquad (9.6)
\end{aligned}$$

Next, observe that $\bigcap_n U_S^{-1}\overline{B(C,\frac{1}{n})}\subseteq U_S^{-1}(\bigcap_n\overline{B(C,\frac{1}{n})})=U_S^{-1}\overline{C}$, whence

$$\begin{aligned}
\sup_{\mu\in\bigcap_n U_S^{-1}\overline{B(C,\frac{1}{n})}}\left(h(\mu)+\int\varphi\,d\mu\right) &\leq \sup_{\mu\in\bigcap_n U_S^{-1}\overline{C}}\left(h(\mu)+\int\varphi\,d\mu\right)\\
&= \sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overline{C}}}\left(h(\mu)+\int\varphi\,d\mu\right). \qquad (9.7)
\end{aligned}$$

Finally, combining (9.5), (9.6) and (9.7) gives inequality (9.2).
(2) This part follows immediately from Part (1) and Proposition 4.2. □

10. Proof of Theorem 4.6

The purpose of this section is to prove Theorem 5.5. We first prove two small lemmas.

Lemma 10.1. *let $\Delta:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ be continuous with $\Delta(\mu)\neq 0$ for all $\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. The either $\Delta<0$ or $\Delta>0$.*

Proof
This is clear since $\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is convex and therefore, in particular, connected. $\square$

Lemma 10.2. *Let X be a normed vector space. Let $\Gamma:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ be continuous and affine and let $\Delta:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ be continuous and affine with $\Delta(\mu)\neq 0$ for all $\mu\in\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})$. Define $U:\mathcal{P}(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to X$ by $U=\frac{\Gamma}{\Delta}$. Let C be a closed and convex subset of X and assume that*

$$\overset{\circ}{C}\cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)\neq\varnothing\,.$$

Then

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in C}}\left(h(\mu)+\int\varphi\,d\mu\right)=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in \overset{\circ}{C}}}\left(h(\mu)+\int\varphi\,d\mu\right).$$

Proof
For brevity define $F:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to\mathbb{R}$ by $F(\mu)=h(\mu)+\int\varphi\,d\mu$. It clearly suffices to show that

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in C}}F(\mu)\leq\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in \overset{\circ}{C}}}F(\mu)\,. \tag{10.1}$$

We will now prove inequality (10.1). Write $s=\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,,\,U\mu\in C}F(\mu)$. Fix $\varepsilon>0$. It follows from the definition of s that we can choose $\lambda\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with $U\lambda\in C$ and $F(\lambda)>s-\varepsilon$. Also, since $\overset{\circ}{C}\cap U\big(\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\big)\neq\varnothing$, we can find $\nu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, with $U\nu\in\overset{\circ}{C}$. For $t\in(0,1)$ we now define $\gamma_t\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ by $\gamma_t=t\nu+(1-t)\lambda$. Next, we prove the following two claims.

Claim 1. For all $t\in(0,1)$, we have $U\gamma_t\in\overset{\circ}{C}$.
Proof of Claim 1. Fix $t\in(0,1)$. Write $a=\frac{t\Delta(\nu)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}$ and $b=\frac{(1-t)\Delta(\lambda)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}$. We now make a few observations. We first observe that it follows from Lemma 10.1 that either $\Delta<0$ or $\Delta>0$. This clearly implies that $a,b\in(0,1)$. Next, we note that $U\gamma_t=\frac{\Gamma(t\nu+(1-t)\lambda)}{\Delta(t\nu+(1-t)\lambda)}=\frac{t\Gamma(\nu)+(1-t)\Gamma(\lambda)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}=\frac{t\Gamma(\nu)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}+\frac{(1-t)\Gamma(\lambda)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}=\frac{t\Delta(\nu)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}U\nu+\frac{(1-t)\Delta(\lambda)}{t\Delta(\nu)+(1-t)\Delta(\lambda)}U\lambda=aU\nu+bU\lambda$. We can now prove that $U\gamma_t\in\overset{\circ}{C}$. Indeed, since $a,b\in(0,1)$ with $a+b=1$ and $U\lambda\in C$ and $U\nu\in\overset{\circ}{C}$, we conclude from [Con, p. 102, Proposition 1.11] that $U\gamma_t=aU\nu+bU\lambda\in\overset{\circ}{C}$. This completes the proof of Claim 1.

Claim 2. There is $\mu_0\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with $U\mu_0\in\overset{\circ}{C}$ such that $F(\mu_0)>s-\varepsilon$.
Proof of Claim 2. Since the entropy function $h:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ is affine (see [Wa]), we conclude that F is affine, and so $F(\gamma_t)=F(t\nu+(1-t)\lambda)=tF(\nu)+(1-t)F(\lambda)\to F(\lambda)>s-\varepsilon$. This implies that there is $t_0\in(0,1)$ with $F(\gamma_{t_0})>s-\varepsilon$. Now put $\mu_0=\gamma_{t_0}$. Then $F(\mu_0)=F(\gamma_{t_0})>s-\varepsilon$ and Claim 1 implies that $U\mu_0=U\gamma_{t_0}\in\overset{\circ}{C}$. This completes the proof of Claim 2.

We can now prove inequality (10.1). Indeed, it follows from Claim 2 that there is $\mu_0\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$ with $U\mu_0\in\overset{\circ}{C}$ such that $F(\mu_0)>s-\varepsilon$, whence $s-\varepsilon<F(\mu_0)\leq\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,,\,U\mu\in\overset{\circ}{C}}F(\mu)$. Finally, letting $\varepsilon\searrow 0$ gives $s\leq\sup_{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\,,\,U\mu\in\overset{\circ}{C}}F(\mu)$. $\square$

We can now prove Theorem 4.6.

Proof of Theorem 4.6
In view of Lemma 10.2, it suffices to prove the following two inequalities, namely,

$$\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overset{\circ}{C}}}\left(h(\mu)+\int\varphi\,d\mu\right)\le\underline{P}_C^U(\varphi)\,, \tag{10.2}$$

$$\overline{P}_C^U(\varphi)\le\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in C}}\left(h(\mu)+\int\varphi\,d\mu\right). \tag{10.3}$$

Proof of inequality (10.2). For $r>0$, let $G_r=\{x\in C\,|\,\mathrm{dist}(x,X\setminus C)>r\}$, and note that G_r is open with $B(G_r,\rho)\subseteq C$ for all $0<\rho<r$. We therefore conclude from Theorem 8.3 and Theorem 8.4 that if $0<\rho<r$, then

$$\begin{aligned}\underline{P}_C^U(\varphi)&\ge\underline{P}_{B(G_r,\rho)}^U(\varphi)\\&\ge\underline{Q}_{G_r}^U(\varphi) &&\text{[by Theorem 8.4]}\\&\ge\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G_r}}\left(h(\mu)+\int\varphi\,d\mu\right). &&\text{[by Theorem 8.3]}\end{aligned} \tag{10.4}$$

Taking supremum over all $r>0$ in (10.4) yields

$$\underline{P}_C^U(\varphi)\ge\sup_{r>0}\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in G_r}}\left(h(\mu)+\int\varphi\,d\mu\right).$$

Letting $U_S:\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\to(0,\infty)$ denote the restriction of U to $\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})$, the previous inequality can be written as

$$\begin{aligned}\underline{P}_C^U(\varphi)&\ge\sup_{r>0}\sup_{\mu\in U_S^{-1}G_r}\left(h(\mu)+\int\varphi\,d\mu\right)\\&=\sup_{\mu\in\bigcup_{r>0}U_S^{-1}G_r}\left(h(\mu)+\int\varphi\,d\mu\right).\end{aligned} \tag{10.5}$$

However, it is easily seen that $\bigcup_{r>0}G_r=\overset{\circ}{C}$, whence $\bigcup_{r>0}U_S^{-1}G_r=U_S^{-1}(\bigcup_{r>0}G_r)=U_S^{-1}\overset{\circ}{C}$. We conclude from this inclusion and inequality (10.5) that

$$\begin{aligned}\underline{P}_C^U(\varphi)&\ge\sup_{\mu\in U_S^{-1}\overset{\circ}{C}}\left(h(\mu)+\int\varphi\,d\mu\right)\\&=\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in\overset{\circ}{C}}}\left(h(\mu)+\int\varphi\,d\mu\right).\end{aligned}$$

This proves inequality (10.2).

Proof of inequality (10.3). Since C is closed we immediately conclude from Theorem 8.3 and Theorem 8.4 that

$$\begin{aligned}\overline{P}_C^U(\varphi)&\le\overline{Q}_C^U(\varphi) &&\text{[by Theorem 8.4]}\\&\le\sup_{\substack{\mu\in\mathcal{P}_S(\Sigma_{\mathsf{G}}^{\mathbb{N}})\\ U\mu\in C}}\left(h(\mu)+\int\varphi\,d\mu\right). &&\text{[by Theorem 8.3]}\end{aligned}$$

This proves inequality (10.3). □

Acknowledgements.

We thank an anonymous referee for detailed and useful comments.

References

ArMa. M. Artin & B. Mazur, *On periodic points*, Ann. of Math. **21** (1965), 82–99.

ArPa. M. Arbeiter & N. Patzschke, *Random self-similar multifractals*, Math. Nachr. **181** (1996), 5–42.

Bak. S. P. Baker, *A multifractal zeta function for Gibbs measures supported on cookie-cutter sets*, Nonlinearity **26** (2013), 1125–1142.

Bal1. V. Baladi, *Dynamical zeta functions and generalized Fredholm determinants.*, XIth International Congress of Mathematical Physics (Paris, 1994), 249–260, Int. Press, Cambridge, MA, 1995.

Bal2. V. Baladi, *A Brief Introduction to Dynamical Zeta Functions*, Classical Nonintegrability, Quantum Chaos DMV Seminar Volume 27, 1997, pp . 3–20.

Bar. L. Barreira, *Thermodynamic Formalism and Applications to Dimension Theory*, Birkhäuser, 2011.

BaMe. J. Barral & M. Mensi, *Multifractal analysis of Birkhoff averages on "self-affine" symbolic spaces*, Nonlinearity **21** (2008), 2409–2425.

Bo1. R. Bowen, *Hausdorff dimension of quasi-circles*, Publications Mathematiques (I.H.E.S. Paris) 50, (1979), 11–26.

Bo2. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.*, Springer Lect. Notes in Math. 470, 1975.

Col1. J. Cole, *The Geometry of Graph Directed Self-Conformal Multifractals*, Ph.D. Thesis, University of St. Andrews, 1998.

Col2. J. Cole, *Relative multifractal analysis*, Chaos, Solitons and Fractals **11** (2000), 2233–2250.

Con. J. Conway, *A Course in Functional Analysis, Second Edition*, Springer Verlag, New York, 1990.

DeZe. A. Dembo & O. Zeitouni, *Large deviations techniques and applications*, Jones and Bartlett Publishers, Boston, MA, 1993.

EdMa. G. A. Edgar & R. D. Mauldin, *Multifractal Decompositions of Digraph Recursive Fractals*, Proc. London Math. Soc. **65** (1992), 604–628.

El. R. Ellis, *Entropy, large deviations, and statistical mechanics. Grundlehren der Mathematischen Wissenschaften, 271*, Springer-Verlag, New York, 1985.

Fa. K. J. Falconer, *Techniques in Fractal Geometry*, Wiley, 1997.

FaFe. A.-H. Fan & D.-J. Feng, *On the distribution of long-term time averages on symbolic space*, J. Statist. Phys. **99** (2000), 813–856.

FaFeWu. A.-H. Fan, D.-J. Feng & J. Wu, *Recurrence, dimension and entropy*, J. London Math. Soc. **64** (2001), 229–244.

FeLaWu. D.-J. Feng, K.-S. Lau & J. Wu, *Ergodic limits on the conformal repellers*, Adv. Math. **169** (2002), 58–91.

Hu. J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

JiQiQi. D.-Q. Jiang, M. Qian & M.-P. Qian, *Mathematical theory of nonequilibrium steady states. On the frontier of probability and dynamical systems*, Lecture Notes in Mathematics, 1833. Springer-Verlag, Berlin, 2004.

LapRo. M. Lapidus & J. Rock, *Towards zeta functions and complex dimensions of multifractals*, Complex Var. Elliptic Equ. **54** (2009), 545559.

LapLe-VeRo. M. Lapidus J. Lévy-Véhel & J. Rock, *Fractal strings and multifractal zeta functions*, Lett. Math. Phys. **88** (2009), 101129.

Lap-vF1. M. L. Lapidus & M. van Frankenhuysen, *Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions*, Birkhäuser, Boston,, 2000.

Lap-vF2. M. L. Lapidus & M. van Frankenhuysen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and spectra of fractal strings*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2006.

Le-VeMe. J. Levy-Vehel & F. Mendivil, *Multifractal and higher-dimensional zeta functions*, Nonlinearity **24** (2011), 259–276.

MaUr. R. D. Mauldin & M. Urbanski, *Graph directed Markov systems. Geometry and dynamics of limit sets*, Cambridge Tracts in Mathematics, 148. Cambridge University Press, Cambridge, 2003.

MiOl. V. Mijović & L. Olsen, *Multifractal spectra and multifractal zeta-functions*, Preprint (2013).

Oli. E. Olivier, *Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures*, Nonlinearity **24** (1999), 1571–1585.

Ol1. L. Olsen, *Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages*, Journal de Mathématiques Pures et Appliquées, **82** (2003), 1591–1649.

Ol2. L. Olsen, *Multifractal tubes*, Preprint (2013).

Ol3. L. Olsen,, *Multifractal tubes: Multifractal zeta-functions, multifractal Steiner formulas and explicit formulas*, "Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics I", Contemporary Mathematics (D. Carfi, M. Lapidus, E. Pearse, et al., editors)..

Ol4. L. Olsen, *Dynamical multifractal zeta-functions, multifractal pressure and fine multifractal spectra*, Preprint (2013).

OlWi. L. Olsen & S. Winter, *Multifractal analysis of divergence points of deformed measure theretical Birkhoff averages. II: non-linearity, divergence points and Banach space valued spectra*, Bulletin des Sciences Mathématiques, **131** (2007), 518–558.

OrPe1. S. Orey & S. Pelikan, *Large deviation principles for stationary processes*, Ann. Probab. **16** (1988), 1481–1495.

OrPe2. S. Orey & S. Pelikan, *Deviations of trajectory averages and the defect in Pesin's formula for Anosov diffeomorphisms*, Trans. Amer. Math. Soc. **315** (1989), 741–753.

ParPo1. W. Parry & M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. **118** (1983), 573591.

ParPo2. W. Parry & M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astrisque No. 187-188 (1990).

Pa. N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math. **19** (1997), 486513.

Pe. Y. Pesin, *Dimension Theory in Dynamical Systems. Contemporary Views and Applications*, The University of Chicago Press, 1997.

Ro. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

Rue1. D. Ruelle, *Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics*, Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley Publishing Co., Reading, Mass., 1978.

Rue2. D. Ruelle, *Dynamical zeta functions for piecewise monotone maps of the interval*, CRM Monograph Series, 4. American Mathematical Society, Providence, RI, 1994.

Va. S. R. S. Varadhan, *Asymptotic probabilities and differential equations*, Comm. Pure Appl. Math. **19** (1966), 261–286.

Wa. P. Walters, *An introduction to ergodic theory. Graduate Texts in Mathematics, Vol. 79*, Springer-Verlag, New York-Berlin, 1982.